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BENDING OF RATE SENSITIVE CIRCULAR PLATES

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "BENDING OF RATE SENSITIVE CIRCULAR PLATES" submitted by TERRY MICHAEL HRUDEY in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

A linearized theory based on piecewise flat level surfaces of the complementary function is developed for an incompressible inelastic dissipative continuum. Uniqueness and extremum principles associated with the constitutive equations derived from the complementary function are discussed.

The theory is applied to the problem of a clamped circular visco-plastic plate loaded with a uniform pressure, and the stationary creep of a plate with the same geometry and loading. Complete solutions are found for the bending moments and rate of deflection for both problems.

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CHAPTER I

INTRODUCTION

1.1 Preliminary

The behavior of axially symmetric flat plates with various edge conditions is of engineering importance and has thus received considerable attention. Until recently however, only the elastic problem had been considered. Recent developments in continuum mechanics have resulted in solutions for this type of plate problem for more complicated constitutive relations.

Hopkins and Prager [9] in 1953 considered the problem of a circular rigid-perfectly plastic plate obeying the Tresca yield condition. Closed form solutions were obtained for the moments and deflection rates along with the corresponding limit loads for both simply supported and clamped edges. The solutions are valid only for small deflections since membrane stresses are not considered. Also the rigid plastic model is more realistic for a sandwich plate than for a solid plate.

The importance of viscous or rate dependent effects in metals at elevated temperatures has resulted in a number of attempts to take these effects into account. This has led to the application of various creep laws and the visco-plastic model to plate problems.

Venkatraman and Hodge [16] have obtained solutions for creep of both simply supported and clamped uniformly loaded circular plates. A creep law based upon a flow rule associated with a condition of maximum shearing stress was assumed. The solution assumes further that stationary creep conditions prevail and that the creep rate can be represented by a function of moment times a function of time. The assumption of stationary creep is equivalent to assuming that deformation occurs under a time invariant state of stress*, since changes in the stress would result in changes in the elastic deformation as well as in additional primary creep.

Prager [15] has suggested a linearized theory of visco-plasticity which involves a piecewise linear yield condition. The proposed model reduces to the Newtonian viscous fluid when the yield stress approaches zero and uniqueness has been proven for the strain rates. This model, with the Tresca yield condition has been applied to the problem of a simply supported circular plate under uniform pressure and a complete solution has been obtained [1]. It appears that the application of this theory to the problem of a clamped plate is intractable.

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* In some literature the term stationary creep is synonymous with secondary creep, that is creep with constant strain rate. In this thesis stationary creep refers to creep occurring under a constant state of stress.

Haddow [3] has suggested the linearized theory discussed in this thesis and has found a complete solution for a visco-plastic simply supported circular plate loaded with a uniform pressure.

This thesis considers the problem of a uniformly loaded clamped circular plate. A linearization technique is developed and applied to two types of isotropic materials. The first is a rigid visco-plastic solid which is a generalization of the Bingham solid.* The second is a material obeying a creep law which is a generalization of the one dimensional law

$$\epsilon = \left(\frac{\sigma}{\sigma_o}\right)^n ,$$

where σ_o and n are material properties and ϵ and σ are the strain rate and the stress respectively [12]. Both materials have as special cases the Newtonian viscous fluid and the rigid-perfectly plastic solid. For n = 1 the creep law above reduces to the constitutive equation in one dimension for a Newtonian viscous fluid and when n approaches infinity the rigid-perfectly plastic solid is approached.

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* The Bingham solid is a solid that has the constitutive equation

$$|\sigma_{12}| = k + 2\mu \, |e_{12}|$$

for a state of simple shear $\sigma_{21} = \sigma_{12}$, where k is the yield stress in pure shear and μ is a coefficient of viscosity [2].

1.2 The Work and Complementary Functions

Hill [5] has attempted to establish a unified theory for several branches of the subject of continuum mechanics. The basis of this unified theory depends on the existence of two convex functions, the work function E and the complementary function E_c.

Let σ_{ij} and $\dot{e}_{ij}$ be the components of the stress and strain rate tensors respectively, referred to the rectangular axes ox_i. If σ_{ij} is a single valued function of e_{ij} then for a purely dissipative medium the stress tensor can be written as

$$\sigma_{ij} = \frac{\partial E}{\partial e_{ij}} \tag{1.1}$$

where for the purpose of differentiation all nine components of e_{ij} are considered to be independent. The function $E(e_{ij})$ is the work function and is defined as

$$E(e_{ij}) = \int \sigma_{ij} \, de_{ij}. \tag{1.2}$$

The condition that E be a convex function is

$$E(e_{ij}^*) - E(e_{ij}) \geqslant (e_{ij}^* - e_{ij}) \frac{\partial E}{\partial e_{ij}} \tag{1.3}$$

for any two strain rate states e_{ij}^* and e_{ij}. Considering a surface of constant E in principal strain rate space*,

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* The concept of principal strain rate space or principal stress space is useful only for isotropic media. For anisotropic media 9 dimensional spaces must be considered.

where the co-ordinate axes are the three principal strain rates, the condition (1.3) requires that the surface is nowhere convex to the origin. If the strict inequality in (1.3) is always valid when e_{ij}^{*} is not equal to e_{ij} then the surface is said to be strictly convex, meaning that a surface of constant E is everywhere concave to the origin and contains no flat surfaces.

The complementary function E_c is defined in a similar manner. If e_{ij} is a single valued function of σ_{ij} then for a purely dissipative medium the components of the strain rate may be written as

$$e_{ij} = \frac{\partial E_c}{\partial \sigma_{ij}} \tag{1.4}$$

where $E_c(\sigma_{ij})$ is called the complementary function and is defined as

$$E_c(\sigma_{ij}) = \int e_{ij}\, d\sigma_{ij} \tag{1.5}$$

The condition for $E_c(\sigma_{ij})$ to be convex is analagous to (1.3) and is

$$E_c(\sigma_{ij}^{*}) - E_c(\sigma_{ij}) \geqslant (\sigma_{ij}^{*} - \sigma_{ij}) \frac{\partial E_c}{\partial \sigma_{ij}} \tag{1.6}$$

for any two stress states σ_{ij} and σ_{ij}^{*}.

From the definitions of E and E_c it follows that the existence of both functions implies a one-one relationship between σ_{ij} and e_{ij} if both E and E_c are strictly convex.

The sum of the work and complementary functions is

$$E + E_c = \int \sigma_{ij}\, de_{ij} + \int e_{ij}\, d\sigma_{ij} = e_{ij}\, \sigma_{ij} \qquad (1.7)$$

and is the rate of energy dissipation per unit volume.

If all nine components of e_{ij} and σ_{ij} are considered as independent variables, equations (1.1), (1.4), and (1.7) are seen to be a Legendre dual transformation [10].

If the strain rate components e_{ij} are taken as coordinates in a nine dimensional space, equation (1.1) indicates that the σ_{ij} are components of a vector normal to a surface of constant E, the components of σ_{ij} being multiplied by a scalar constant to give them the dimensions of strain. Similarly from equation (1.4) the e_{ij} are seen to be the components of a vector normal to a surface of constant E_c in nine dimensional stress space.

If the material is incompressible, the first invariant of the strain rate tensor, e_{kk}, is zero. Consequently $\frac{\partial E_c}{\partial \sigma_{kk}}$ is zero and E_c is independent of the hydrostatic component of stress σ_{kk}. Thus the stress tensor σ_{ij} can be replaced by its deviator s_{ij}, where

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\, \sigma_{kk}\, \delta_{ij} \, . \qquad (1.8)$$

An example of an incompressible material for which the work and complementary functions exist is the classical Newtonian viscous fluid. The two convex functions for this material are

$$E = \mu e_{ij} e_{ij} = \mu \frac{I}{2} \tag{1.9}$$

$$E_c = \frac{s_{ij} s_{ij}}{4\mu} = \frac{J}{2\mu} \tag{1.10}$$

where $I = 2e_{ij} e_{ij}$ and $J = \frac{1}{2} s_{ij} s_{ij}$ are invariants of the strain rate and stress deviator tensors respectively, and μ is the viscosity. The constitutive relation which may be derived from either (1.9) or (1.10) is

$$e_{ij} = \frac{s_{ij}}{2\mu} . \tag{1.11}$$

Equation (1.10) is a special case of a more general complementary function of the form

$$E_c = f(J) . \tag{1.12}$$

If the principal stresses σ_1, σ_2, and σ_3 are taken as rectangular co-ordinate axes, a surface of constant E_c given by (1.12) is a circular cylinder whose axis makes equal angles with each of the co-ordinate axes.

A further example of a material with a complementary function of the form given by (1.12) is the generalization of the Bingham solid* suggested by Prager [14] and Hohenemser and Prager [7], for which

$$E_c = \left\langle \sqrt{J} - K \right\rangle^2 / 2\mu \tag{1.13}$$

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* Often in the literature this solid is referred to as the Bingham solid. Actually it is a generalization of the one dimensional model suggested by Bingham. Henceforth it will be referred to in this thesis as the generalized Bingham solid.

and $$E = \mu \frac{I}{2} + K \sqrt{I} \qquad I \neq 0 \tag{1.14}$$

where μ is the viscosity and K is the yield stress in pure shear for the Mises yield condition. The angle brackets have the following significance.

$$\begin{aligned} \langle F \rangle &= 0 & \text{for } F < 0 \\ &= F & \text{for } F \geqslant 0 \end{aligned}$$

The constitutive equations derivable from the work and complementary functions by use of (1.1) and (1.4) are

$$s_{ij} = 2(\mu + \frac{K}{\sqrt{I}})\, e_{ij} \qquad I \neq 0 \tag{1.15}$$

and $$e_{ij} = \frac{1}{2\mu} \left\langle 1 - \frac{K}{\sqrt{J}} \right\rangle s_{ij} \,. \tag{1.16}$$

From (1.15) it follows that the stress in the generalized Bingham solid subjected to a strain rate e_{ij} is the sum of the stresses in a Newtonian viscous fluid and a Mises rigid plastic solid each subjected to the same strain rate.

The work function (1.14) is equal to one half the rate of viscous energy dissipation per unit volume plus the total rate of plastic energy dissipation per unit volume. The complementary function (1.13) is equal to one half the rate of viscous energy dissipation per unit volume.

The complementary and work functions for both the Newtonian viscous fluid and the Bingham solid are both strictly convex and thus the uniqueness theorems due to Hill are valid.

1.3 Linearization

The basic hypothesis of the classical inviscid theory of plasticity [4] is the existence of a yield condition $f(\sigma_{ij}) = c$ and the equivalence of the yield function f with the plastic potential. The plastic strain rates e^p_{ij} for a rigid-perfectly plastic material are thus given by

$$e^p_{ij} = \lambda \frac{\partial f}{\partial s_{ij}} \tag{1.17}$$

where λ is a non-negative parameter and

$$\lambda = 0 \text{ if } f < c \quad \text{or} \quad \text{if } f = 0 \text{ and } \dot{f} < 0$$
$$\lambda \geqslant 0 \text{ if } f = c \quad \text{and} \quad \dot{f} = 0$$

where $\dot{f}$ is the material derivative of f.

In principal stress space the vector with components (e^p_1, e^p_2, e^p_3) is normal to the yield surface $f(\sigma_{ij}) = c$. For the Mises yield condition

$$f(\sigma_{ij}) = \frac{1}{2} s_{ij} s_{ij} \tag{1.18}$$

and $c = k^2$ where k is the yield stress in pure shear. The yield surface is a circular cylinder whose axis passes through the origin and makes equal angles with each of the co-ordinate axes.

A similarity in form can thus be seen between the flow rule for a Mises rigid plastic solid and a material with a complementary function given by (1.12).

The Tresca yield condition is given by

$$\left|\sigma_{max} - \sigma_{min}\right| = 2k \tag{1.19}$$

where σ_{max} and σ_{min} are the maximum and minimum principal stresses respectively and k is the yield stress in pure shear. The yield surface for a Tresca solid is a regular hexagon in principal stress space. It may further be regarded as a piecewise linear approximation for the Mises yield surface.

This suggests the possibility of a similar piecewise approximation for a complementary function of the form (1.12). Such an approximation* could be given in general as

$$E_c = g(\sigma) \tag{1.20}$$

where

$$\sigma = \left|\sigma_{max} - \sigma_{min}\right| . \tag{1.21}$$

By a choice of the proper function $g(\sigma)$, the surface of constant E_c given by (1.20) could be made to fit inside the surface of constant E_c given by (1.12), such that the two surfaces co-incide only at the corners of the hexagonal prism (1.21).

This approach when applied to the Newtonian viscous

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* Although this is an approximation in the sense that a cylindrical surface of constant E_c is replaced by a hexagonal prism in principal stress space, there is experimental evidence [17] that the constitutive equations derived from the latter form of the E_c surface represent more closely the creep behavior of some materials. This is analagous to the fact that the yield condition for most real metals lies somewhere between the Mises and the Tresca yield conditions.

fluid yields the following form of the complementary function.

$$E_c = \frac{(\sigma_{max} - \sigma_{min})^2}{6\mu} \tag{1.22}$$

The intersection of the two surfaces of constant E_c given by (1.10) and (1.22) with the plane σ_3 = constant, in principal stress space, are shown in Fig. 1.1. The position of the point (σ_3, σ_3) is determined by the hydrostatic component of stress.

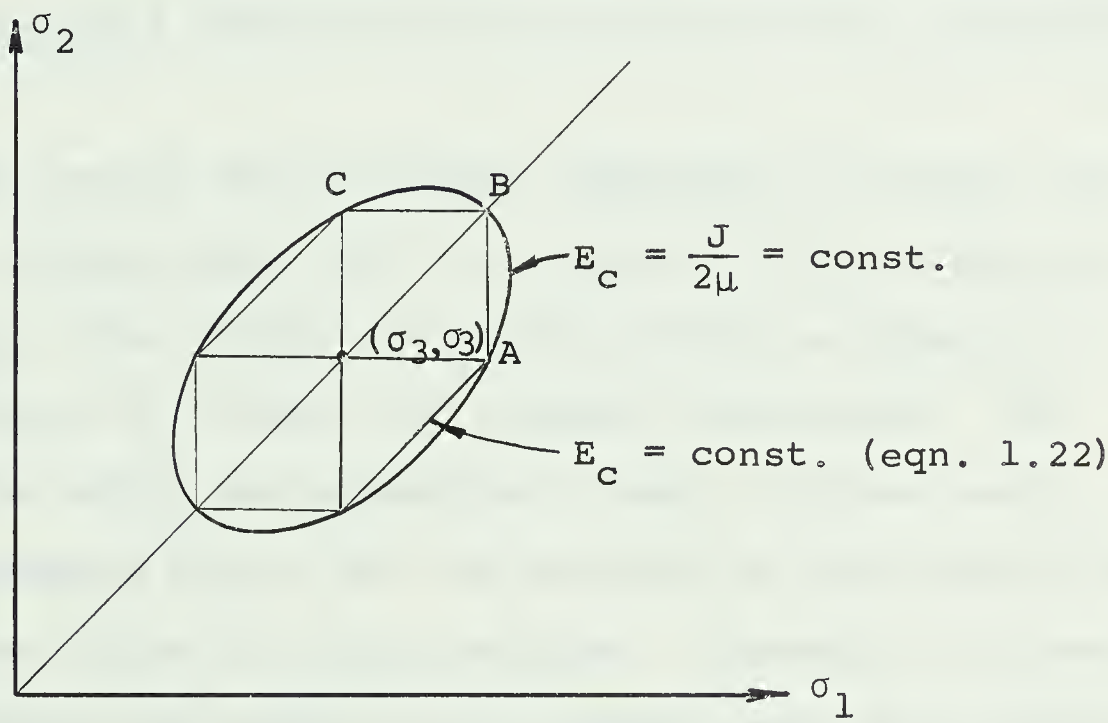

FIGURE 1.1

SURFACES OF CONSTANT E_c FOR VISCOUS FLUID

If the stress point lies on a flat of the surface of constant E_c, the strain rate vector corresponding to that state of stress is normal to the flat. Thus, if the three principal stresses are distinct, meaning that the stress

point is on a flat, the principal components of the strain rate obtained from equation (1.4) are

$$e_{max} = (\sigma_{max} - \sigma_{min})/3\mu = -e_{min}, \quad e_{int} = 0 \qquad (1.23)$$

where e_{max}, e_{int}, and e_{min} are the maximum, intermediate, and minimum principal components. Since this is true for any point on the flat the components of the stress tensor are not single valued functions of the components of the strain rate for a material with this form of the complementary function.

If any two of the principal components of stress are equal the stress point lies on a corner of the surface of constant E_c. The strain rate vector which is normal to the surface of E_c is thus not uniquely determined. The strain rate vector corresponding to such a stress state, can lie anywhere within the fan enclosed by the normals to the adjacent flats of the E_c surface. Although the direction of the strain rate vector is not unique there is a restriction on the magnitude of the vector. The three principal components of the strain rate vector for any vector lying in the fan enclosed by the normals to the two adjacent flats of the E_c surface, must satisfy the condition that the rate of energy dissipation per unit volume is constant for any point on the E_c surface. This follows from the fact that for this material, E_c is one half of the total rate of energy

dissipation per unit volume. This condition may be stated as

$$\sigma_1 e_1 + \sigma_2 e_2 + \sigma_3 e_3 = \text{constant} . \tag{1.24}$$

Referring to Fig. 1.1, the strain rate vector for a point on the flat AB has components

$$e_1 = \frac{(\sigma_1 - \sigma_3)}{3\mu} = -e_3 \;, \quad e_2 = 0 \tag{1.25}$$

and the strain rate vector for a stress point on the flat CB has components

$$e_1 = 0, \quad e_2 = \frac{(\sigma_2 - \sigma_3)}{3\mu} = -e_3 \;. \tag{1.26}$$

The strain rate vector for point B can thus be found using (1.24), (1.25), and (1.26). It has the components

$$e_1 = \frac{t(\sigma_1 - \sigma_3)}{3\mu}, \quad e_2 = \frac{(1-t)(\sigma_2 - \sigma_3)}{3\mu}, \quad e_3 = -e_1 - e_2 \tag{1.27}$$

where t is a parameter which can take any value in the closed interval $0 \leqslant t \leqslant 1$. Similar expressions may be found for each point on the E_c surface.

It should be noted that the strain rates resulting from this approximation of E_c are independent of the intermediate principal stress. This is also true for the Tresca yield condition and its associated flow rule.

This linearization technique can be applied to materials with more complicated constitutive relations. Consider now

a material in which the stress is given by the sum of the stresses in a Tresca rigid-perfectly plastic solid and a viscous fluid with a complementary function given by (1.22), when all three bodies are subjected to the same strain rate. This model is a generalization of the Bingham solid. The complementary function for this material is found to be

$$E_c = \frac{\left\langle \sigma_{max} - \sigma_{min} - 2k \right\rangle^2}{6\mu} \tag{1.28}$$

where μ is a coefficient of viscosity and k is the yield stress in pure shear for the Tresca yield condition. Again the surface of constant E_c is a regular hexagonal prism in principal stress space.

In a manner similar to that used previously for the viscous fluid, the strain rates may be obtained. For a stress point on a flat of the E_c surface the principal strain rate components are

$$e_{max} = \frac{\left\langle \sigma_{max} - \sigma_{min} - 2k \right\rangle}{3\mu} = -e_{min}, \quad e_{int} = 0 \; . \tag{1.29}$$

As before, the strain rate vector for a stress point on a corner of the E_c surface must lie in the fan enclosed by the normals to the two adjacent flats, with the restriction (1.24) on the magnitude of the strain rate vector.

Fig. 1.2 shows the intersection of the surfaces of constant E_c given by (1.13) and (1.28), and the plane σ_3 = constant, along with the intersection of the Tresca

and Mises yield surfaces with this plane.

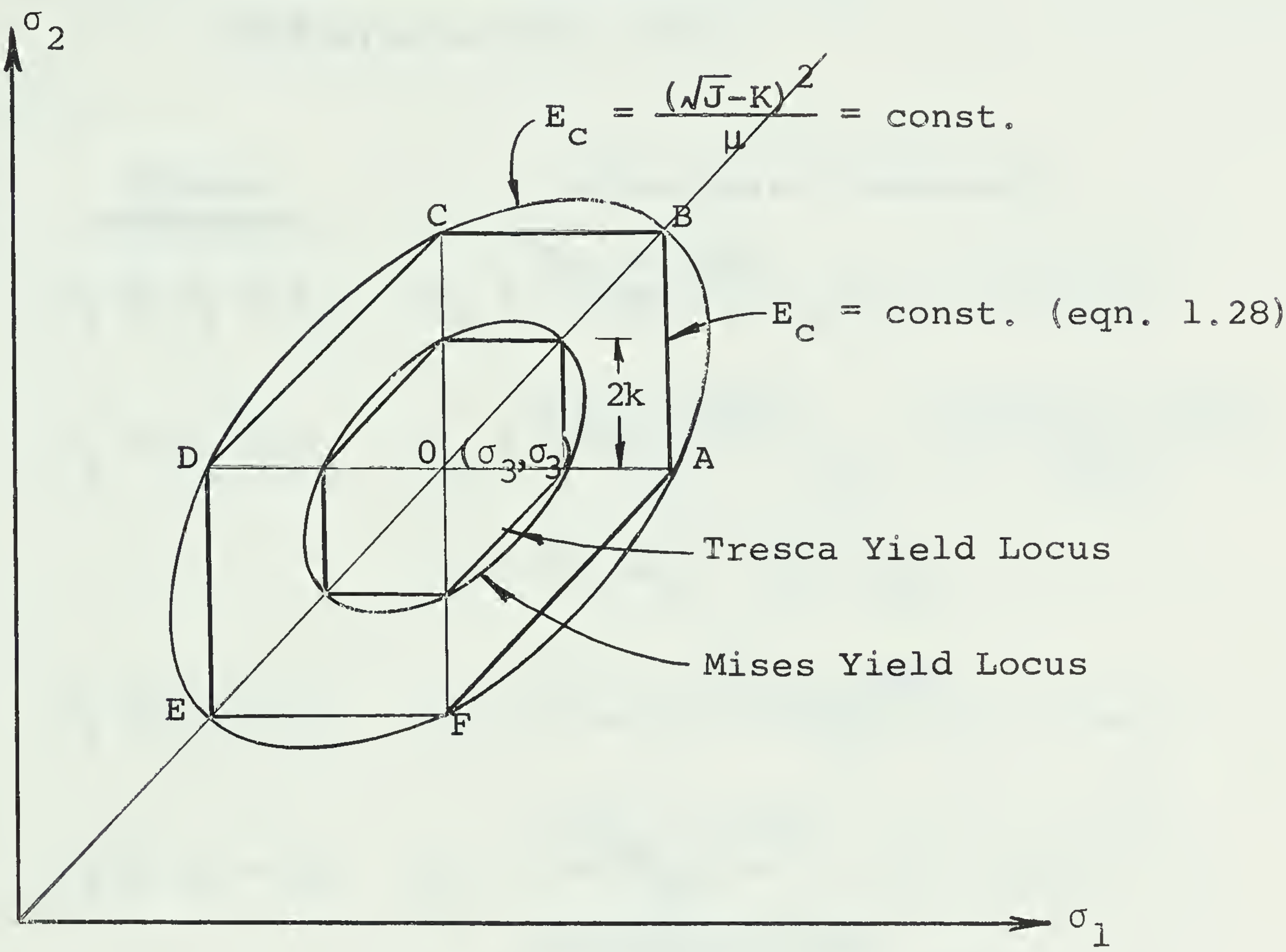

FIGURE 1.2

SURFACES OF CONSTANT E_c FOR VISCO-PLASTIC MATERIAL

Table I gives the strain rates corresponding to any stress state for this material and thus constitutes a complete flow rule.

TABLE I

FLOW RULE FROM EQUATION (1.29)

(Referred to Fig. 1.2)

Stress Regime	Stress Components	Strain Rate Components
BOA	$\sigma_1 > \sigma_2 > \sigma_3$	$e_1 = \frac{\langle\sigma_1-\sigma_3-2k\rangle}{3\mu}$, $e_2 = 0$, $e_3 = -e_1$
OB	$\sigma_1 = \sigma_2 > \sigma_3$	$e_1 = \frac{t\langle\sigma_1-\sigma_3-2k\rangle}{3\mu}$, $e_2 = \frac{(1-t)\langle\sigma_2-\sigma_3-2k\rangle}{3\mu}$, $e_3 = -e_1-e_2$, $0 \leqslant t \leqslant 1$
BOC	$\sigma_2 > \sigma_1 > \sigma_3$	$e_1 = 0$, $e_2 = \frac{\langle\sigma_2-\sigma_3-2k\rangle}{3\mu}$, $e_3 = -e_2$
OC	$\sigma_2 > \sigma_1 = \sigma_3$	$e_1 = \frac{-s\langle\sigma_2-\sigma_1-2k\rangle}{3\mu}$, $e_2 = -e_3-e_1$, $e_3 = \frac{(s-1)\langle\sigma_2-\sigma_3-2k\rangle}{3\mu}$, $0 \leqslant s \leqslant 1$
COD	$\sigma_2 > \sigma_3 > \sigma_1$	$e_1 = -e_2$, $e_2 = \frac{\langle\sigma_2-\sigma_1-2k\rangle}{3\mu}$, $e_3 = 0$
OD	$\sigma_2 = \sigma_3 > \sigma_1$	$e_1 = -e_2-e_3$, $e_2 = \frac{(1-q)\langle\sigma_2-\sigma_1-2k\rangle}{3\mu}$, $e_3 = \frac{q\langle\sigma_3-\sigma_1-2k\rangle}{3\mu}$, $0 \leqslant q \leqslant 1$
DOE	$\sigma_3 > \sigma_2 > \sigma_1$	$e_1 = -e_3$, $e_2 = 0$, $e_3 = \frac{\langle\sigma_3-\sigma_1-2k\rangle}{3\mu}$

Stress Regime	Stress Components	Strain Rate Components
OE	$\sigma_3 > \sigma_2 = \sigma_1$	$e_1 = \frac{(m-1)\langle\sigma_3-\sigma_1-2k\rangle}{3\mu}$, $e_2 = \frac{-m\langle\sigma_3-\sigma_2-2k\rangle}{3\mu}$, $e_3 = -e_2-e_1$, $0 \leqslant m \leqslant 1$
EOF	$\sigma_3 > \sigma_1 > \sigma_2$	$e_1 = 0$, $e_2 = -e_3$, $e_3 = \frac{\langle\sigma_3-\sigma_2-2k\rangle}{3\mu}$
OF	$\sigma_3 = \sigma_1 > \sigma_2$	$e_1 = \frac{g\langle\sigma_1-\sigma_2-2k\rangle}{3\mu}$, $e_2 = -e_3-e_1$, $e_3 = \frac{(g-1)\langle\sigma_3-\sigma_2-2k\rangle}{3\mu}$, $0 \leqslant g \leqslant 1$
FOA	$\sigma_1 > \sigma_3 > \sigma_2$	$e_1 = \frac{\langle\sigma_1-\sigma_2-2k\rangle}{3\mu}$, $e_2 = -e_1$, $e_3 = 0$
OA	$\sigma_1 > \sigma_3 = \sigma_2$	$e_1 = -e_2-e_3$, $e_2 = \frac{(f-1)\langle\sigma_1-\sigma_2-2k\rangle}{3\mu}$, $e_3 = \frac{-f\langle\sigma_1-\sigma_3-2k\rangle}{3\mu}$, $0 \leqslant f \leqslant 1$

CHAPTER II

UNIQUENESS AND EXTREMUM PRINCIPLES

2.1 Uniqueness

The classical boundary value problem for a dissipative inelastic medium is posed as follows. Determine the distribution of stress and velocity if the surface traction F_i is prescribed on part S_f of the surface of a body and the velocity v_i is prescribed on the part S_v of the surface.* The uniqueness of the solution for such a problem depends on the nature of the constitutive relationship for the material.

The constitutive equations of the continua which are considered in this thesis, except for the rigid-perfectly plastic solid, are obtained from the complementary function E_c. The two types of continua considered have as special cases the Tresca rigid-perfectly plastic solid and the viscous fluid with a complementary function given by (1.22).

The uniqueness theorem for a rigid plastic solid has been given by Hill [7]**. If (σ_{ij}, v_i) and $(\overset{*}{\sigma}_{ij}, \overset{*}{v}_i)$ are two solutions for the rigid plastic boundary value problem, then by the virtual work equation

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* Surfaces S_f and S_v may overlap. Then the force (velocity) normal to the surface and the velocity (force) tangent to the surface are prescribed in the overlapping region.

** This uniqueness theorem and the corresponding extremum principles are valid for a rigid plastic workhardening solid provided the current distribution of hardening is known.

$$\int_S (F_i - F_i^*)(v_i - v_i^*)\,dS = \int_V (\sigma_{ij} - \sigma_{ij}^*)(e_{ij} - e_{ij}^*)\,dV + \int_{S_D} (k - \tau^*)\left[v\right] dS_D$$

$$+ \int_{S_D^*} (k - \tau)\left[v^*\right] dS_D^* \qquad (2.1)$$

where $\left[v\right]$ and $\left[v^*\right]$ are the velocity discontinuities on S_D and S_D^* respectively and τ^* and τ are the shear stresses corresponding to σ_{ij}^* and σ_{ij} acting on S_D and S_D^* in the directions of the velocity discontinuities $\left[v^*\right]$ and $\left[v\right]$ respectively.

To show uniqueness it is necessary to introduce the maximum work inequality for a rigid plastic material,

$$(\sigma_{ij} - \sigma_{ij}^*)\, e_{ij} \geqslant 0 \qquad (2.2)$$

where e_{ij} is the strain rate corresponding to the stress σ_{ij} in a plastically deforming body and σ_{ij}^* is any stress point lying on or within the current yield surface. If the yield surface is strictly convex then the equality in (2.2) can hold only if σ_{ij} and σ_{ij}^* are equal. If the yield surface contains flats then the equality in equation (2.2) is true only if the two stress points lie on the same flat of the yield surface.

Since $F_i = F_i^*$ on S_f and $v_i = v_i^*$ on S_v, the left hand side of (2.1) must equal zero. From the maximum work inequality and the inequalities

$$k \geqslant \tau, \qquad k \geqslant \tau^*$$

where k is the yield stress in pure shear, it follows that

each term on the right hand side of (2.1) is positive and therefore each integrand is zero. That is

$$(\sigma_{ij} - \sigma^*_{ij})\, e_{ij} = 0 \quad \text{and} \quad (\sigma^*_{ij} - \sigma_{ij})\, e^*_{ij} = 0 \,. \qquad (2.3)$$

Thus if the yield surface is strictly convex, equations (2.2) and (2.3) indicate that the stress field is unique except perhaps in the common non-deforming region of the two solutions. If the yield surface has flats then equations (2.2) and (2.3) require that σ_{ij} and σ^*_{ij} be on the same flat everywhere except perhaps in the common non-deforming region. It follows that the stress solution in the deforming region is not necessarily unique. The velocity solution is not unique regardless of whether the yield surface has flats or not.

In the axially symmetric Tresca rigid-perfectly plastic plate problem, which is a special case of the two problems in Chapters III and IV, an a priori assumption is made as to the stress regimes in which the stress solution lies and a solution is obtained [13]. The uniqueness theorem for a Tresca rigid plastic material indicates that all stress solutions must lie on the same stress profile. It is found that for any particular stress profile, only one stress solution at most can be found which satisfies the equilibrium equation, the flow rule, and the boundary conditions. Thus the stress solution which has been found elsewhere for the circular clamped Tresca rigid-perfectly plastic plate problem is unique [9].

It remains now to investigate the boundary value problem for a material with a constitutive relation of the general form

$$e_{ij} = \frac{\partial E_c}{\partial \sigma_{ij}} . \tag{2.4}$$

If the stress in the material is dependent on the time rate of deformation then velocity discontinuities cannot occur since infinite stresses would be required to produce such discontinuities. Therefore the virtual work equation becomes

$$\int_V (\sigma_{ij} - \sigma^*_{ij})(e_{ij} - e^*_{ij})\, dV = 0 \tag{2.5}$$

where (σ_{ij}, e_{ij}) and $(\sigma^*_{ij}, e^*_{ij})$ are two solutions to the boundary value problem.

Substitution of the constitutive equation (2.4) into (2.5) gives

$$\int_V (\sigma_{ij} - \sigma^*_{ij}) \left(\frac{\partial E_c(\sigma_{ij})}{\partial \sigma_{ij}} - \frac{\partial E_c(\sigma^*_{ij})}{\partial \sigma^*_{ij}} \right) dV = 0. \tag{2.6}$$

Equation (2.6) is satisfied if

$$(\sigma_{ij} - \sigma^*_{ij}) \frac{\partial E_c(\sigma_{ij})}{\partial \sigma_{ij}} = (\sigma_{ij} - \sigma^*_{ij}) \frac{\partial E_c(\sigma^*_{ij})}{\partial \sigma^*_{ij}} . \tag{2.7}$$

From the condition of convexity for the E_c surface the following inequality is obtained

$$(\sigma_{ij} - \sigma^*_{ij}) \frac{\partial E_c(\sigma_{ij})}{\partial \sigma_{ij}} \geqslant E_c(\sigma_{ij}) - E_c(\sigma^*_{ij}) \geqslant (\sigma_{ij} - \sigma^*_{ij}) \frac{\partial E_c(\sigma^*_{ij})}{\partial \sigma^*_{ij}} . \tag{2.8}$$

If the surface of constant E_c is strictly convex then the equality in (2.8) holds only if σ_{ij} and σ^*_{ij} are equal. It follows therefore from (2.6) and (2.7) that the stress solution is unique. Furthermore, since E_c is strictly convex there is a one-one relationship between stress and strain rate and thus the strain rate is also unique.

If E_c is not strictly convex* then the equation (2.7) and the equality in (2.8) are satisfied if σ_{ij} and σ^*_{ij} lie on the same flat of the surface of constant E_c. This implies that the stress profile is unique.

The materials considered in the two plate problems in this thesis have complementary functions whose surfaces are hexagonal prisms in principal stress space. For these problems the stress profile is thus unique. As for the Tresca rigid-perfectly plastic plate problems it is found that for any stress profile only one solution can be found which satisfies the equilibrium equation, flow rule, and the boundary conditions. It follows that the complete solution for a plate problem with a constitutive relation defined in terms of the complementary function is unique.

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* The surfaces of constant E_c considered in this discussion are assumed not to be convex to the origin at any point. Thus the distinction between a strictly convex, and a convex E_c surface is that the latter may have flats.

2.2 Extremum Principles

Two extremum principles may be derived for a dissipative material for which the complementary and work functions exist. They have been presented but not proved by Hill [5].

If (σ_{ij}, v_i) are the solution stress and velocity fields for this classical boundary value problem and σ^*_{ij} is any statically admissible stress field then it follows from the virtual work equation that

$$\int_S (F_i - F^*_i)\ v_i\ dS = \int_V (\sigma_{ij} - \sigma^*_{ij})\ e_{ij}\ dV. \tag{2.9}$$

But $F_i = F^*_i$ on S_f and from the convexity of the complementary function

$$(\sigma_{ij} - \sigma^*_{ij})\ e_{ij} \geqslant E_c(\sigma_{ij}) - E_c(\sigma^*_{ij})\ .$$

Thus equation (2.9) becomes

$$\int_{S_v} F^*_i\ v_i\ dS - \int_V E_c(\sigma^*_{ij})\ dV \leqslant \int_{S_v} F_i\ v_i\ dS - \int_V E_c(\sigma_{ij})\ dV \tag{2.10}$$

which is the first extremum principle.

If (σ_{ij}, v_i) are the solution stress and velocity fields and v^*_i is any kinematically admissible velocity field for this boundary value problem then from the virtual work equation

$$\int_S F_i(v^*_i - v_i)\ dS + \int_V X_i(v^*_i - v_i)\ dV = \int_V \sigma_{ij}(e^*_{ij} - e_{ij})\,dV \tag{2.11}$$

where X_i is the body force per unit volume.

On S_v, $v_i^* = v_i$, and from the convexity of the work function

$$(e_{ij}^* - e_{ij})\,\sigma_{ij} \leqslant E(e_{ij}^*) - E(e_{ij}) \;.$$

Thus equation (2.11) becomes the second extremum principle

$$\int_V \left(E(e_{ij}^*) - X_i\, v_i^*\right) dV - \int_{S_f} F_i\, v_i^*\, dS \geqslant \int_V \left(E(e_{ij}) - X_i\, v_i\right) dV - \int_{S_f} F_i\, v_i\, dS \qquad (2.12)$$

The right hand sides of both (2.10) and (2.12) are necessarily unique.

CHAPTER III

RIGID VISCO-PLASTIC PLATE PROBLEM

Let r, θ, and z be cylindrical co-ordinates with z taken positive downwards. Consider a plate bounded by the surfaces $z = \pm h/2$ and $r = a$.

The plate is clamped along the edge $r = a$ and is loaded with a uniform pressure p acting in the positive z direction. The rigid plastic or rigid visco-plastic model is realistic for a sandwich type construction consisting of two thin outer layers which carry stresses in the plane of the plate only, and an inner layer which is capable of carrying vertical shear only. When bending occurs these outer layers are in a bi-axial state of stress. Since membrane stresses, resulting from extension of the middle surface of the plate, are assumed small in comparison to the bending stresses and since the stress is essentially constant across the thickness of these outer layers, the bending moments per unit length M_r and M_θ are proportional to the stresses in the outer layers. Thus the generalized stresses M_r and M_θ are used in the following discussion rather than σ_r and σ_θ.

Let w be the velocity of any point in the middle surface of the plate, taken positive in the positive z direction. The usual assumptions in small deflection plate theory give the curvature rates in the radial and circumferential directions as

$$\kappa_r = -\frac{d^2w}{dr^2} \quad \text{and} \quad \kappa_\theta = -\frac{1}{r}\frac{dw}{dr} \,. \tag{3.1}$$

The curvature rates and the moments are taken to be positive if they correspond to compression in the $z = -h/2$ surface. These curvature rates may be regarded as generalized strain rates.

Consider now an element of the plate bounded by the circles with radii r and $r+dr$ and two radial lines containing an elemental angle $d\theta$. Let Q be the vertical shear stress per unit length acting on the surfaces r = constant as shown in Fig. 3.1.

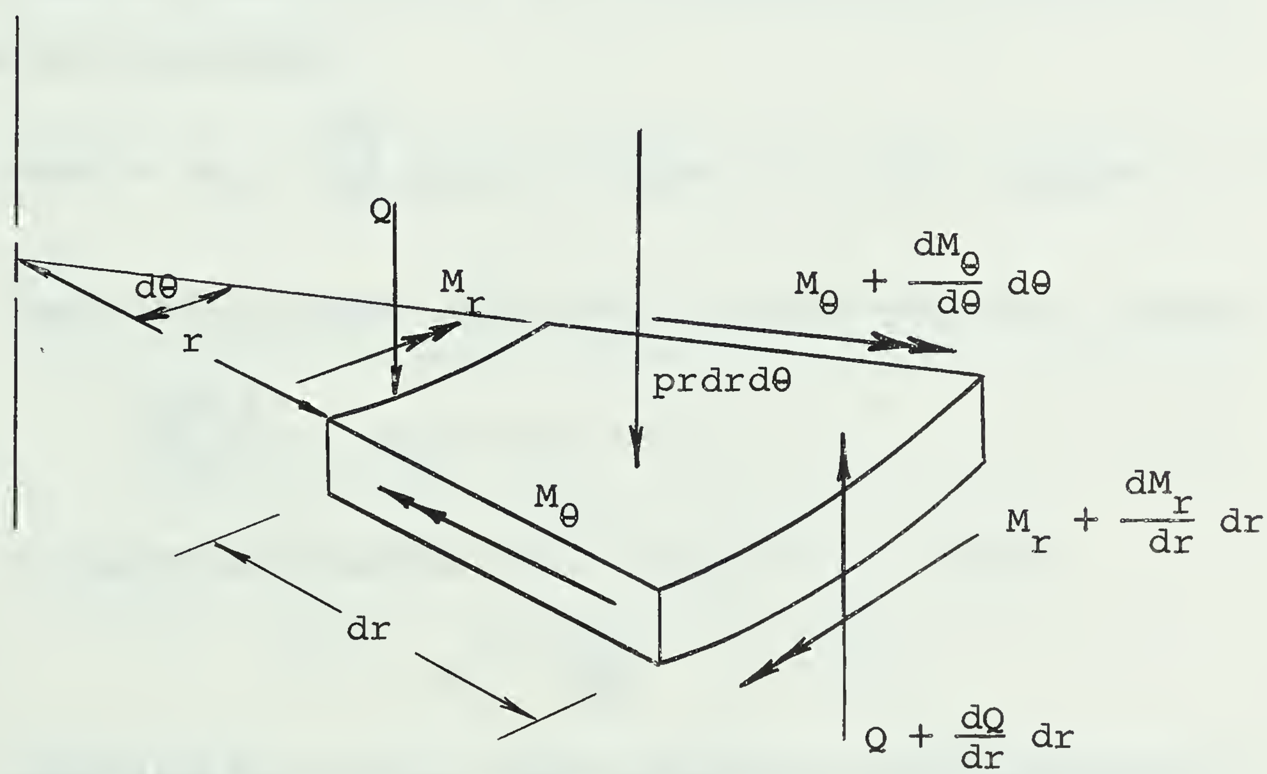

FIGURE 3.1

PLATE ELEMENT

Summing forces in the z direction results in the following equilibrium equation

$$+ prdrd\theta - (Q + \frac{dQ}{dr} dr)(r + dr)d\theta + Qrd\theta = 0 . \qquad (3.2)$$

Neglecting products of differentials of degree greater than two yields the following equation,

$$\frac{d(rQ)}{dr} - pr = 0 \qquad (3.3)$$

Equilibrium of the moments acting about a radial line through the element is automatically satisfied due to the circular symmetry of the problem.

Summing moments about the circumferential direction gives the following,

$$-M_\theta drd\theta + (M_r + \frac{dM_r}{dr} dr)(r + dr)d\theta - M_r rd\theta + Qrdrd\theta = 0. \qquad (3.4)$$

Neglecting higher order small quantities this reduces to

$$\frac{d(rM_r)}{dr} - M_\theta + Qr = 0 . \qquad (3.5)$$

Integrating equation (3.3) from 0 to r gives

$$Q = \frac{pr}{2} . \qquad (3.6)$$

Substitution of this into equation (3.5) yields the complete equilibrium condition for the plate in terms of the generalized stresses M_r and M_θ.

$$\frac{d(rM_r)}{dr} - M_\theta + \frac{pr^2}{2} = 0 \qquad (3.7)$$

The constitutive relation which is to be applied to this plate problem is defined in terms of a complementary function of the form (1.28) and may be regarded as an approximation to the generalized Bingham solid. The complementary function may be expressed in terms of the generalized stresses M_r and M_θ as

$$E_c = \frac{\langle M - M_0 \rangle^2}{2\lambda} \tag{3.8}$$

where $M = \max(|M_r|, |M_\theta|, |M_r - M_\theta|)$ and λ is a coefficient of viscosity whose relation to μ depends on the plate geometry.

The complementary function given by equation (3.8) is equal to half the rate of viscous energy dissipation per unit surface area of the plate. If the middle surfaces of the two outer layers of thickness t are separated by a distance h then $\lambda = 3\mu t^2 h^2$. Similarly the yield moment M_0 is related to k, the yield stress in pure shear according to $M_0 = 2kth$.

The curvature rates can be found by partial differentiation of the complementary function with respect to the moments M_r and M_θ. This results in the flow rule

$$\kappa_r = \frac{\partial E_c}{\partial M_r} \quad \text{and} \quad \kappa_\theta = \frac{\partial E_c}{\partial M_\theta} \quad . \tag{3.9}$$

Referring to Fig. 3.2 which shows a surface of constant E_c given by equation (3.8), the flow rule in Table II is obtained from equations (3.8) and (3.9).

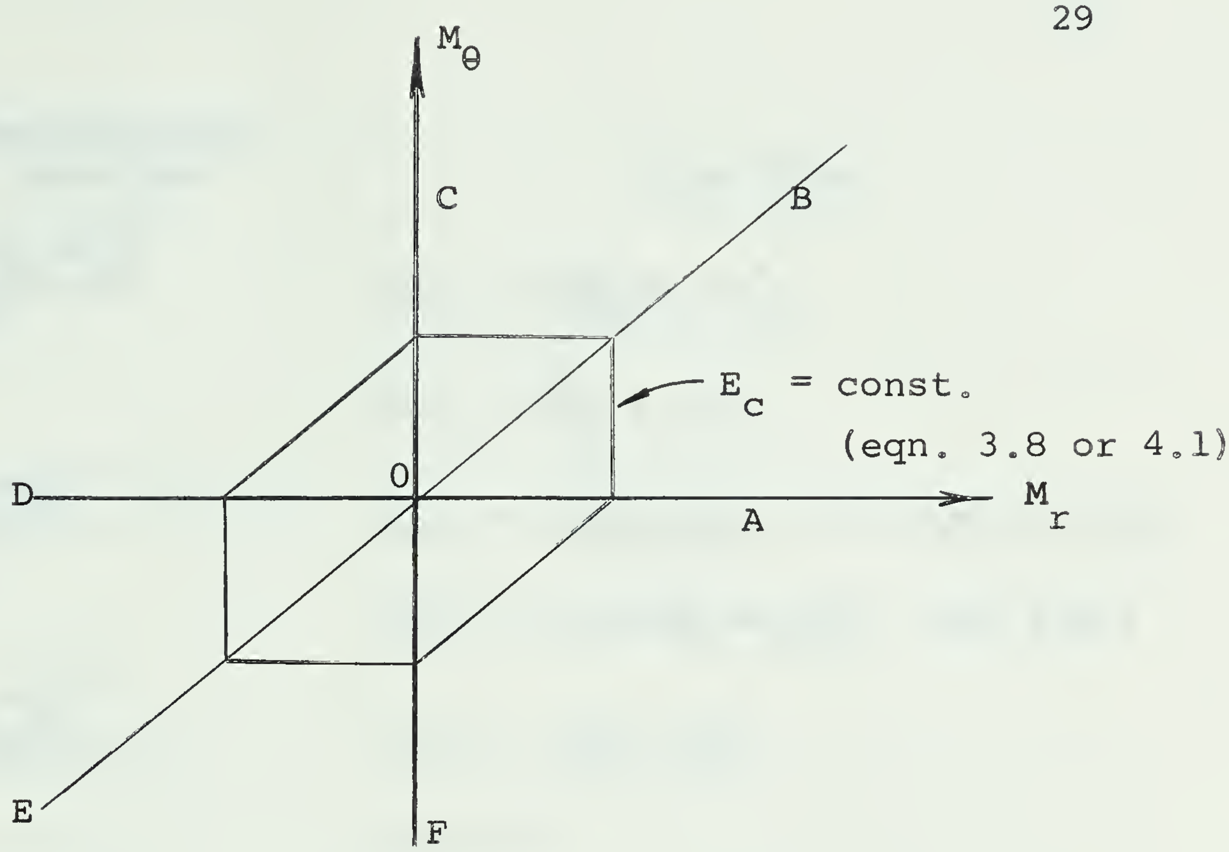

FIGURE 3.2

SURFACE OF CONSTANT E_c

TABLE II

FLOW RULE FROM EQUATIONS (3.8) AND (3.9)

Stress Regime	Complementary Function	Flow Rule
COB	$\dfrac{\langle M_\theta - M_0\rangle^2}{2\lambda}$	$\lambda\kappa_r = 0$ $\lambda\kappa_\theta = \langle M_\theta - M_0\rangle$
OC	$\dfrac{\langle M_\theta - M_0\rangle^2}{2\lambda}$	$\lambda\kappa_r = -s\langle M_\theta - M_r - M_0\rangle \qquad 0 \leqslant s \leqslant 1$ $\lambda\kappa_\theta = (1-s)\langle M_\theta - M_0\rangle + s\langle M_\theta - M_r - M_0\rangle$

Stress Regime	Complementary Function	Flow Rule
COD	$\dfrac{\langle M_\theta - M_r - M_0\rangle^2}{2\lambda}$	$\lambda\kappa_r = -\langle M_\theta - M_r - M_0\rangle$ $\lambda\kappa_\theta = \langle M_\theta - M_r - M_0\rangle$
OD	$\dfrac{\langle -M_r - M_0\rangle^2}{2\lambda}$	$\lambda\kappa_r = -t\langle -M_r - M_0\rangle + (t-1)\langle M_\theta - M_r - M_0\rangle$ $\lambda\kappa_\theta = (1-t)\langle M_\theta - M_r - M_0\rangle \quad 0 \leqslant t \leqslant 1$
DOE	$\dfrac{\langle -M_r - M_0\rangle^2}{2\lambda}$	$\lambda\kappa_r = -\langle -M_r - M_0\rangle$ $\lambda\kappa_\theta = 0$
OE	$\dfrac{\langle -M_r - M_0\rangle^2}{2\lambda} = \dfrac{\langle -M_\theta - M_0\rangle^2}{2\lambda}$	$\lambda\kappa_r = -q\langle -M_r - M_0\rangle$ $\lambda\kappa_\theta = (q-1)\langle -M_\theta - M_0\rangle \quad 0 \leqslant q \leqslant 1$
EOF	$\dfrac{\langle -M_\theta - M_0\rangle^2}{2\lambda}$	$\lambda\kappa_r = 0$ $\lambda\kappa_\theta = -\langle -M_\theta - M_0\rangle$
OF	$\dfrac{\langle -M_\theta - M_0\rangle^2}{2\lambda}$	$\lambda\kappa_r = m\langle M_r - M_\theta - M_0\rangle \quad 0 \leqslant m \leqslant 1$ $\lambda\kappa_\theta = (m-1)\langle -M_\theta - M_0\rangle - m\langle M_r - M_\theta - M_0\rangle$
FOA	$\dfrac{\langle M_r - M_\theta - M_0\rangle^2}{2\lambda}$	$\lambda\kappa_r = \langle M_r - M_\theta - M_0\rangle$ $\lambda\kappa_\theta = -\langle M_r - M_\theta - M_0\rangle$
OA	$\dfrac{\langle M_r - M_0\rangle^2}{2\lambda}$	$\lambda\kappa_r = g\langle M_r - M_\theta - M_0\rangle + (1-g)\langle M_r - M_0\rangle$ $\lambda\kappa_\theta = -g\langle M_r - M_\theta - M_0\rangle \quad 0 \leqslant g \leqslant 1$

Stress Regime	Complementary Function	Flow Rule
AOB	$\dfrac{\langle M_r - M_0\rangle^2}{2\lambda}$	$\lambda\kappa_r = \langle M_r - M_0\rangle$ $\lambda\kappa_\theta = 0$
OB	$\dfrac{\langle M_r - M_0\rangle^2}{2\lambda} = \dfrac{\langle M_\theta - M_0\rangle^2}{2\lambda}$	$\lambda\kappa_r = f\langle M_r - M_0\rangle$ $\lambda\kappa_\theta = (1-f)\langle M_\theta - M_0\rangle \quad 0 \leqslant f \leqslant 1$

A complete solution of this plate problem consists of a stress solution, M_r and M_θ, satisfying the equilibrium equation (3.7) and the stress boundary conditions, and a velocity solution w(r) which satisfies the velocity boundary conditions and is derivable from the stress solution by use of equations (3.1) and the flow rule of Table II.

The boundary conditions for this problem are

$$\frac{dw}{dr}(a) = 0 \tag{3.10}$$

$$\frac{dw}{dr}(0) = 0 \tag{3.11}$$

$$w(a) = 0 \tag{3.12}$$

These differ from those for the corresponding Tresca rigid-perfectly plastic plate problem [13]. In the latter problem it is possible to admit a non-zero slope at the plate edge due to the formation of a hinge circle. A hinge circle is a line along which κ_θ is discontinuous. Along a hinge circle $|M_r|$ equals M_0. If a sufficient number of hinge

circles form it is kinematically possible for all the bending to occur at the hinge circles only and for the plate to collapse*. This is analagous to the formation of plastic hinges in a beam or frame. In the visco-plastic plate problem, discontinuities in $\frac{dw}{dr}$ and thus κ_θ are not possible since due to the viscous behavior of the material, infinite stresses would be required to produce such discontinuities. Consequently $\frac{dw}{dr}$ is continuous throughout the visco-plastic plate.

The boundary conditions, the equilibrium equation, and the flow rule constitute the complete boundary value problem. If the entire plate lies in one stress regime then the problem is relatively simple. If however the stress solution covers more than one stress region then each region with its corresponding flow rule must be considered separately.

To begin the solution an a priori assumption is made as to the stress profile. For this problem it is clear that the stress profile includes more than one stress regime since M_r must change sign at some point in the plate.

From circular symmetry M_r and M_θ must be equal at the centre of the plate and since compression occurs in the top surface of the plate at $r = 0$ when it deflects downward,

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* Plastic collapse of a rigid-perfectly plastic structure occurs at that point in the loading program at which deformation first becomes possible. The corresponding load is called the limit load.

both M_r and M_θ are positive at that point. It is reasonable to assume therefore that in some finite region around $r = 0$ the stress points would lie in stress regime OB. Knowing also that M_r must be negative at $r = a$, the stress profile in Table III is assumed.

TABLE III

ASSUMED STRESS PROFILE FOR VISCO-PLASTIC PLATE PROBLEM

Plate Region	Stress Regime
$0 \leqslant r \leqslant e$	OB
$e \leqslant r \leqslant d$	BOC
$d \leqslant r \leqslant b$	COD
$b \leqslant r \leqslant a$	OD

The flow rule for each of these regions in the plate are given in Table IV.

TABLE IV

FLOW RULE FOR VISCO-PLASTIC PLATE PROBLEM

Plate Region	Curvature Rates	
$0 \leqslant r \leqslant e$	$\lambda\kappa_r = f\langle M_r - M_0\rangle \quad \lambda\kappa_\theta = (1-f)\langle M_\theta - M_0\rangle$	$0 \leqslant f \leqslant 0.5$ $f = 0.5$ at $r = 0$ $f = 0$ at $r = e$
$e \leqslant r \leqslant d$	$\lambda\kappa_r = 0 \quad \lambda\kappa_\theta = \langle M_\theta - M_0\rangle$	
$d \leqslant r \leqslant b$	$\lambda\kappa_r = -\langle M_\theta - M_r - M_0\rangle \quad \lambda\kappa_\theta = \langle M_\theta - M_r - M_0\rangle$	
$b \leqslant r \leqslant a$	$\lambda\kappa_r = -t\langle -M_r - M_0\rangle + (t-1)\langle M_\theta - M_r - M_0\rangle \quad \lambda\kappa_\theta = (1-t)\langle M_\theta - M_r - M_0\rangle$	$0 \leqslant t \leqslant 1$ $t = 1$ at $r = a$

Each of these four regions in the plate are now considered separately.

In the first region, $0 \leqslant r \leqslant e$, the two bending moments are equal and positive. Addition of the curvature rates gives

$$\lambda(\kappa_r+\kappa_\theta) = -\lambda\left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) = -\frac{\lambda}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) = n(M_r-M_0)+(1-n)(M_\theta-M_0) .$$

Since $M_r = M_\theta$ the above becomes

$$-\frac{\lambda}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) = M_\theta - M_0 . \qquad (3.13)$$

The equilibrium equation reduces to

$$\frac{dM_r}{dr} = -\frac{pr}{2} .$$

Integration gives*

$$M_r = -\frac{pr^2}{2} + c_1 = M_\theta . \qquad 0 \leqslant r \leqslant e$$

Substitution of this expression for M_θ into equation (3.13) gives

$$\frac{\lambda}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) = -\frac{pr^2}{2} + c_1 - M_0 .$$

Integrating this result and making use of boundary condition (3.11) gives an expression for the velocity.

$$\lambda w = \frac{pr^4}{64} + \frac{M_0 r^2}{4} - \frac{c_1 r^2}{4} + c_2 \qquad 0 \leqslant r \leqslant e$$

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* The subscripted c's are all constants of integration.

In the second region of the plate, $e \leqslant r \leqslant d$, the curvature rate κ_r is seen to be zero. Therefore $\frac{d^2w}{dr^2} = 0$. Integrating this twice results in

$$w = c_3 r + c_4 . \qquad e \leqslant r \leqslant d$$

Using this result and the relation (3.1) yields

$$\lambda\kappa_\theta = M_\theta - M_0 = -\frac{\lambda}{r}\frac{dw}{dr} = -\frac{\lambda c_3}{r} .$$

Consequently

$$M_\theta = -\frac{\lambda c_3}{r} + M_0 . \qquad e \leqslant r \leqslant d$$

Substitution of this result into the equilibrium equation and integrating yields

$$M_r = -\frac{pr^2}{6} + M_0 - \frac{\lambda c_3}{r} \ln r + \frac{c_5}{r} . \qquad e \leqslant r \leqslant d$$

In the third region of the plate, $d \leqslant r \leqslant b$, the curvature rates are equal in magnitude and opposite in sign, consequently

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} = 0 .$$

This is an Euler type differential equation and is solved by making the substitution $r = e^t$. The solution is

$$w = c_6 \ln r + c_7 \qquad d \leqslant r \leqslant b$$

Substituting this into the flow rule for this region gives

$$\kappa_\theta = -\frac{\lambda}{r}\frac{dw}{dr} = -\frac{\lambda c_6}{r^2} = M_\theta - M_r - M_0 . \tag{3.14}$$

Solving this for M_θ and substituting into the equilibrium

equation yields

$$r \frac{dM_r}{dr} + M_r - \left(- \frac{\lambda c_6}{r^2} + M_r + M_0 \right) + \frac{pr^2}{2} = 0 .$$

Integrating this result gives an expression for M_r from which M_θ can be determined using (3.14). Thus

$$M_r = M_0 \ln r - \frac{pr^2}{4} + \frac{\lambda c_6}{2r^2} + c_8$$

$$d \leqslant r \leqslant b$$

$$M_\theta = M_0(1 + \ln r) - \frac{pr^2}{4} - \frac{\lambda c_6}{2r^2} + c_8 .$$

In the outermost region of the plate, $b \leqslant r \leqslant a$, the bending moment M_θ is zero. From this and the flow rule it follows that

$$\lambda \kappa_r = - \lambda \frac{d^2 w}{dr^2} = M_r + M_0 . \qquad (3.15)$$

Setting M_θ to zero in the equilibrium equation and integrating results in

$$M_r = - \frac{pr^2}{6} - \frac{c_9}{r} . \qquad b \leqslant r \leqslant a$$

Combining this with equation (3.15) and integrating gives

$$\lambda \frac{dw}{dr} = - M_0 r + \frac{pr^3}{18} + c_9 \ln r + c_{10} . \qquad (3.16)$$

Boundary condition (3.10) determines the value of c_{10}.

$$c_{10} = M_0 a - \frac{pa^3}{18} - c_9 \ln a$$

Integration of (3.16) and use of boundary condition (3.12) gives

$$\lambda w = - \frac{M_0}{2} (r-a)^2 + \frac{p}{72} (r^4 - 4a^3 r + 3a^4) + c_9 (r \ln \frac{r}{a} + a - r) .$$

$$b \leqslant r \leqslant a$$

For each of the four regions in the plate an expression has been found for the two moments M_r and M_θ, and for the velocity w. It remains now to eliminate the nine constants of integration.

From equilibrium considerations M_r must be a continuous function of the radius. Similarly M_θ, w, and $\frac{dw}{dr}$ are also continuous functions of r. Therefore at each of the three boundaries separating the four regions of the plate, these continuity conditions provide four equations.

At r = e continuity of w, $\frac{dw}{dr}$, M_r, and M_θ yields the following equations respectively.

$$\frac{pe^2}{64} - \frac{c_1 e^2}{4} + \frac{M_0 e^2}{4} + c_2 = \lambda c_3 e + \lambda c_4 \tag{3.17}$$

$$\frac{pe^3}{16} - \frac{c_1 e}{2} + \frac{M_0 e}{2} = \lambda c_3 \tag{3.18}$$

$$-\frac{pe^2}{4} + c_1 = -\frac{\lambda c_3}{e} \ln e + M_0 - \frac{pe^2}{6} + \frac{c_5}{e} \tag{3.19}$$

$$-\frac{pe^2}{4} + c_1 = -\frac{\lambda c_3}{e} + M_0 \tag{3.20}$$

At the boundary r = d the continuity conditions result in the following four equations.

$$c_3 d + c_4 = c_6 \ln d + c_7 \tag{3.21}$$

$$c_3 = c_6/d \tag{3.22}$$

$$-\frac{\lambda c_3}{d} \ln d + M_0 - \frac{pd^2}{6} + \frac{c_5}{d} = \frac{\lambda c_6}{2d^2} + M_0 \ln d - \frac{pd^2}{4} + c_8 \tag{3.23}$$

$$- \frac{\lambda c_3}{d} + M_0 = - \frac{\lambda c_6}{2d^2} + M_0(1 + \ln d) - \frac{pd^2}{4} + c_8 \qquad (3.24)$$

The continuity conditions at $r = b$ result in another four equations.

$$\lambda c_6 \ln b + \lambda c_7 = \frac{-M_0}{2}(b-a)^2 + \frac{p}{72}(b^4 - 4a^3 b + 3a^4)$$
$$+ c_9(b \ln \frac{b}{a} + a - b) \qquad (3.25)$$

$$\frac{\lambda c_6}{b} = - M_0(b-a) + \frac{p}{18}(b^3 - a^3) + c_9 \ln \frac{b}{a} \qquad (3.26)$$

$$\frac{\lambda c_6}{2b^2} + M_0 \ln b - \frac{pb^2}{4} + c_8 = - \frac{pb^2}{6} - \frac{c_9}{b} \qquad (3.27)$$

$$0 = \frac{-\lambda c_6}{2b^2} + M_0(1 + \ln b) - \frac{pb^2}{4} + c_8 \qquad (3.28)$$

Thus there are a total of twelve independent simultaneous algebraic equations from which the twelve unknowns can be found. The unknowns are the nine constants of integration and the three boundary radii e, d, and b.

The nine constants of integration are first solved for in terms of the other three unknowns.

Solving equation (3.20) for c_1 and substituting the result into equation (3.18) gives

$$c_3 = - \frac{pe^3}{8\lambda}$$

and $$c_1 = \frac{3pe^3}{8} + M_0 .$$

From equation (3.22)

$$c_6 = \frac{-pe^3 d}{8\lambda}$$

Solving equation (3.19) for c_5 and substitution of c_1 and c_3 gives

$$c_5 = \frac{pe^3}{24} (7 - 3 \ln e) .$$

Substitution of c_6 into equation (3.28) results in

$$c_8 = \frac{p}{16b^2} (4b^4 - e^3 d) - M_0 (1 + \ln d) .$$

The constant c_9 may be found in terms of c_6 and c_8 from equation (3.27).

$$c_9 = \frac{p}{24b} (3e^3 d - 4b^4) + M_0 b$$

Solving equation (3.25) for c_7 results in

$$c_7 = \frac{M_0}{\lambda} \left(\frac{-a^2}{2} + 2ab - \frac{3b^2}{2} + b^2 \ln \frac{b}{a} \right) + \frac{p}{\lambda} \left[e^3 \frac{d \ln b}{8} + \frac{b^4}{72} - \frac{a^3 b}{18} + \frac{a^4}{24} + \left(\frac{e^3 d}{8} - \frac{b^4}{6} \right) \left(\ln \frac{b}{a} + \frac{a}{b} - 1 \right) \right] .$$

Substitution of c_3, c_6, and c_7 into equation (3.21) yields

$$c_4 = \frac{M_0}{\lambda} \left(- \frac{a^2}{2} + 2ab - \frac{3b^2}{2} + b^2 \ln \frac{b}{a} \right) + \frac{p}{\lambda} \left[\frac{13b^4}{72} + \frac{e^3 d}{8} \ln \frac{b}{d} - \frac{a^3 b}{18} + \frac{a^4}{24} + \left(\frac{e^3 d}{8} - \frac{b^4}{6} \right) \left(\ln \frac{b}{a} + \frac{a}{b} \right) \right] .$$

Finally from equation (3.17) an expression can be found for c_2.

$$c_2 = M_0\left(-\frac{a^2}{2} + 2ab - \frac{3b^2}{2} + b^2 \ln\frac{b}{a}\right) + p\left[-\frac{3e^4}{64} + \frac{13b^4}{72}\right.$$

$$\left. - \frac{a^3b}{18} + \frac{e^3d}{8}\ln\frac{b}{d} + \frac{a^4}{24} + \left(\frac{e^3d}{8} - \frac{b^4}{6}\right)\left(\ln\frac{b}{a} + \frac{a}{b}\right)\right] .$$

All nine constants of integration have thus been found in terms of the radii e, d, and b. There remain three equations, (3.23), (3.24), and (3.26), which have not been used. Substitution of the expressions for the constants c_1 to c_9 into these three equations results in three trans-endental equations from which e, d, and b may be determined.

The following non-dimensional quantities are now introduced.

$$\rho = \frac{r}{a}, \quad \epsilon = \frac{e}{a}, \quad \delta = \frac{d}{a}, \quad \beta = \frac{b}{a}, \quad A = \frac{M_0}{pa^2}$$

Substitution for c_3, c_5, c_6, and c_8 into equation (3.23) and use of the above non-dimensional parameters results in the first of the three equations.

$$A\left(2 - \ln\frac{\delta}{\beta}\right) + \frac{\epsilon^3}{8\delta}\left(\frac{17}{6} - \ln\frac{\epsilon}{\delta}\right) + \frac{\delta^2}{12} - \frac{\beta^4}{4} + \frac{\epsilon^3\delta}{16\beta^2} = 0 \qquad (3.29)$$

The other two equations which are found from equations (3.24) and (3.26) are

$$16A\left(1 - \ln\frac{\delta}{\beta}\right) + \frac{\epsilon^3}{\delta} + 4\delta^2 - 4\beta^2 + \frac{\epsilon^3\delta}{\beta^2} = 0 \qquad (3.30)$$

and
$$-\frac{\epsilon^3\delta}{8\beta}\left(1 + \ln\beta\right) - \frac{\beta^3}{18}\left(1 - 3\ln\beta\right) + \frac{1}{18} + A\left[-1 + \beta\left(1 - \ln\beta\right)\right] = 0 . \qquad (3.31)$$

The quantity $A = \frac{M_0}{pa^2}$ will be called the loading parameter since it is a measure of the loading on the plate relative to the yield strength. Hopkins and Prager [9] have shown that for a Tresca rigid-perfectly plastic clamped circular plate loaded with a uniform pressure p, deformation does not occur until the pressure p reaches the critical value $p_{cr} = \frac{11.26\ M_0}{a^2}$, then the plate collapses. The rigid-perfectly plastic plate cannot support a pressure greater than this critical value without undergoing finite deformation which would result in membrane stresses. A visco-plastic plate however can support loads greater than p_{cr} due to viscous stresses. Therefore in the visco-plastic plate problem the loading parameter A can take values from 1/11.26, corresponding to the special case of a rigid-perfectly plastic plate, to the value zero representing a visco-plastic plate with zero yield moment.

Hopkins and Prager found also that for the Tresca rigid-perfectly plastic plate the boundaries between the stress regions are

$$\epsilon = 0, \quad \delta = 0.730, \quad \beta = 1.0$$

Substitution of these values of ϵ, δ, and β into equations (3.29), (3.30), and (3.31) along with A = 1/11.26 shows that this solution thus far reduces to that found by Hopkins and Prager for the special case of the rigid-perfectly plastic plate.

Solutions for ϵ, δ, and β have been found for a number of values of A ranging from 1/11.26 to zero. These results have been found with the aid of a computer using an iterative method based on Newton's method for obtaining the roots of an equation [11]. The variation of the three stress region boundaries with the parameter A is shown in Fig. 3.3.

By substituting the solutions for the nine constants c_1 to c_9 into the equations for w, M_r, and M_θ for each of the four stress regions, a complete solution to this problem is obtained, since for any value of the loading parameter A the stress and velocity solutions are completely determined with the aid of Fig. 3.3.

The stress solution thus obtained is

$$\left.\begin{aligned} \frac{M_r}{M_0} &= 1 + \frac{1}{8A}\left[3\epsilon^2 - 2\rho^2\right] \\ \frac{M_\theta}{M_0} &= 1 + \frac{1}{8A}\left[3\epsilon^2 - 2\rho^2\right] \end{aligned}\right\} \quad 0 \leqslant \rho \leqslant \epsilon$$

$$\left.\begin{aligned} \frac{M_r}{M_0} &= 1 + \frac{1}{A}\left[\frac{7\epsilon^3}{24\rho} - \frac{\rho^2}{6} - \frac{\epsilon^3 \ln \epsilon/\rho}{8\rho}\right] \\ \frac{M_\theta}{M_0} &= 1 + \frac{1}{A}\left[\epsilon^3/8\rho\right] \end{aligned}\right\} \quad \epsilon \leqslant \rho \leqslant \delta$$

$$\left.\begin{aligned} \frac{M_r}{M_0} &= \ln\frac{\rho}{\beta} - 1 + \frac{1}{A}\left[\frac{1}{4}(\beta^2 - \rho^2) - \frac{\epsilon^3\delta}{16}\left(\frac{1}{\rho^2} + \frac{1}{\beta^2}\right)\right] \\ \frac{M_\theta}{M_0} &= \ln\frac{\rho}{\beta} + \frac{1}{A}\left[\frac{1}{4}(\beta^2 - \rho^2) + \frac{\epsilon^3\delta}{16}\left(\frac{1}{\rho^2} - \frac{1}{\beta^2}\right)\right] \end{aligned}\right\} \quad \delta \leqslant \rho \leqslant \beta$$

$$\left.\begin{aligned}\frac{M_r}{M_0} &= -\frac{\beta}{\rho} + \frac{1}{A}\left[-\frac{\rho^2}{6} - \frac{\epsilon^3\delta}{8\beta\rho} + \frac{\beta^3}{6\rho}\right] \\ \frac{M_\theta}{M_0} &= 0 \;.\end{aligned}\right\}\quad \beta \leqslant \rho \leqslant 1.0$$

The complete velocity solution is

$$\frac{w\lambda}{M_0 a^2} = \left[-\frac{1}{2} + 2\beta - \frac{3\beta^2}{2} + \beta^2 \ln \beta\right] + \frac{1}{A}\left[\frac{\rho^4 - 6\epsilon^2\rho^2 - 3\epsilon^4}{64} + \frac{13\beta^4}{72} + \frac{\epsilon^3\delta}{8} \ln \frac{\beta}{\delta} - \frac{\beta}{18} + \frac{1}{24} + \left(\frac{\epsilon^3\delta}{8} - \frac{\beta^4}{6}\right)\left(\ln \beta + \frac{1}{\beta}\right)\right] \qquad 0 \leqslant \rho \leqslant \epsilon$$

$$\frac{w\lambda}{M_0 a^2} = \left[-\frac{1}{2} + 2\beta - \frac{3\beta^2}{2} + \beta^2 \ln \beta\right] + \frac{1}{A}\left[-\frac{\epsilon^3\rho}{8} + \frac{\epsilon^3\delta}{8} \ln \frac{\beta}{\delta} + \frac{13\beta^4}{72} - \frac{\beta}{18} + \frac{1}{24} + \left(\frac{\epsilon^3\delta}{8} - \frac{\beta^4}{6}\right)\left(\ln \beta + \frac{1}{\beta}\right)\right] \qquad \epsilon \leqslant \rho \leqslant \delta$$

$$\frac{w\lambda}{M_0 a^2} = \left[-\frac{1}{2} + 2\beta - \frac{3\beta^2}{2} + \beta^2 \ln \beta\right] + \frac{1}{A}\left[-\frac{\epsilon^3\delta}{8} \ln \frac{\rho}{\beta} + \frac{\beta^4}{72} - \frac{\beta}{18} + \frac{1}{24} + \left(\frac{\epsilon^3\delta}{8} - \frac{\beta^4}{6}\right)\left(\ln \beta + \frac{1}{\beta} - 1\right)\right] \qquad \delta \leqslant \rho \leqslant \beta$$

$$\frac{w\lambda}{M_0 a^2} = \left[\beta(1 - \rho) + \beta\rho \ln \rho - \frac{1}{2}(\rho - 1)^2\right] + \frac{1}{A}\left[\frac{\rho^4}{72} - \frac{\rho}{18} + \frac{1}{24} + \left(\frac{\epsilon^3\delta}{8\beta} - \frac{\beta^3}{6}\right)\left(\rho \ln \rho + 1 - \rho\right)\right] . \qquad \beta \leqslant \rho \leqslant 1.0$$

When the viscous stresses in the plate become very small in comparison with the stresses causing plastic

deformation this solution approaches that found by Hopkins and Prager [9] for the corresponding Tresca rigid-perfectly plastic plate.

It may be verified that parameters f and t as defined in Table IV lie in the following closed intervals.

$$0 \leqslant f \leqslant 0.5 \quad \text{and} \quad 0 \leqslant t \leqslant 1$$

Also it may be shown that,

$$f = 0.5 \quad \text{at} \quad r = 0 \qquad t = 0 \quad \text{at} \quad r = b$$

$$f = 0 \quad \text{at} \quad r = e \qquad t = 1 \quad \text{at} \quad r = a.$$

Numerical results for the stress and velocity solution have been obtained for values of A ranging from 1/11.26 to zero and are plotted in Figures (3.3), (3.4), (3.5), (3.6), (3.7) and (3.8).

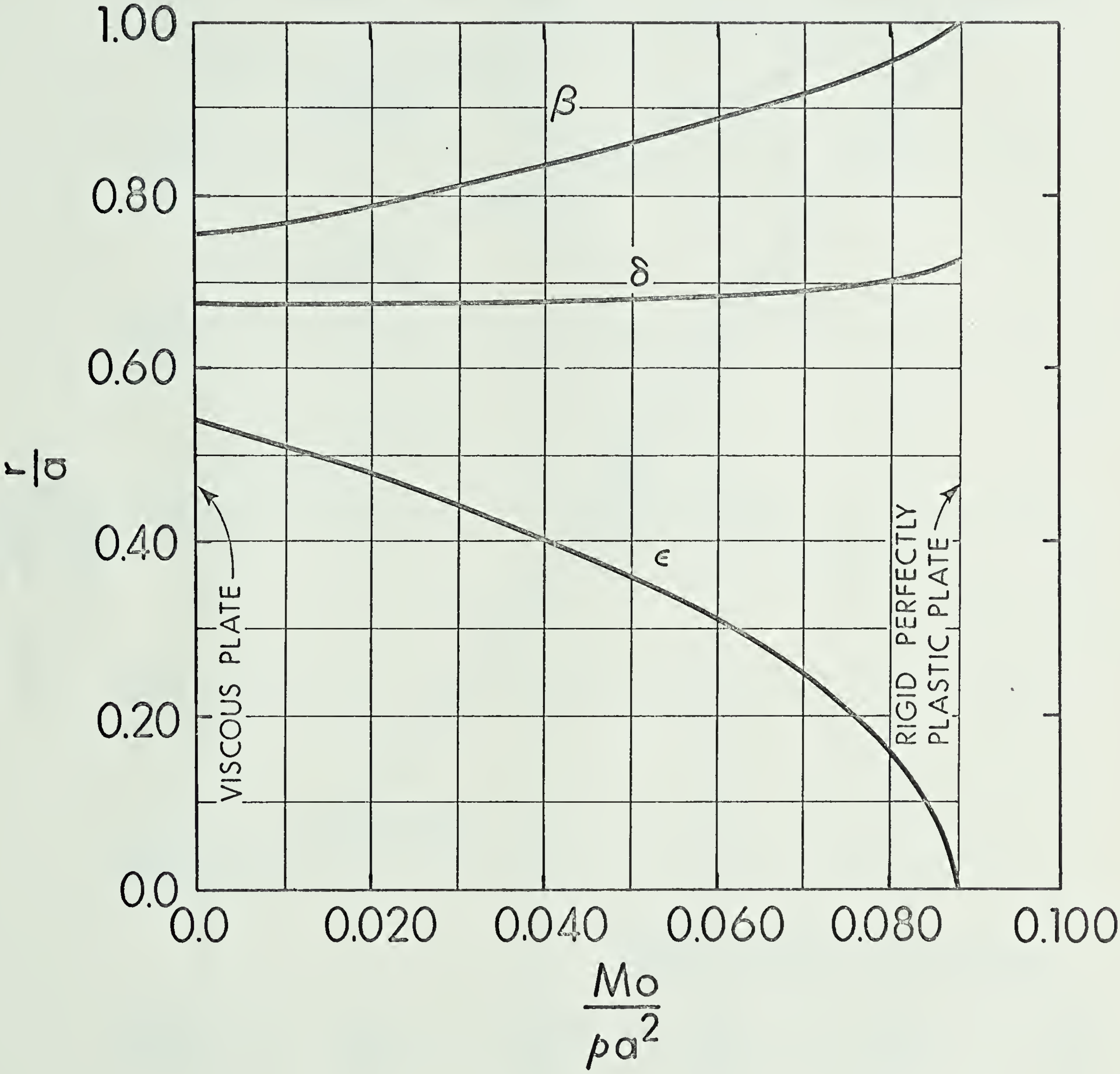

FIG. 3.3

VISCO-PLASTIC PLATE

STRESS REGION BOUNDARIES vs. LOADING PARAMETER

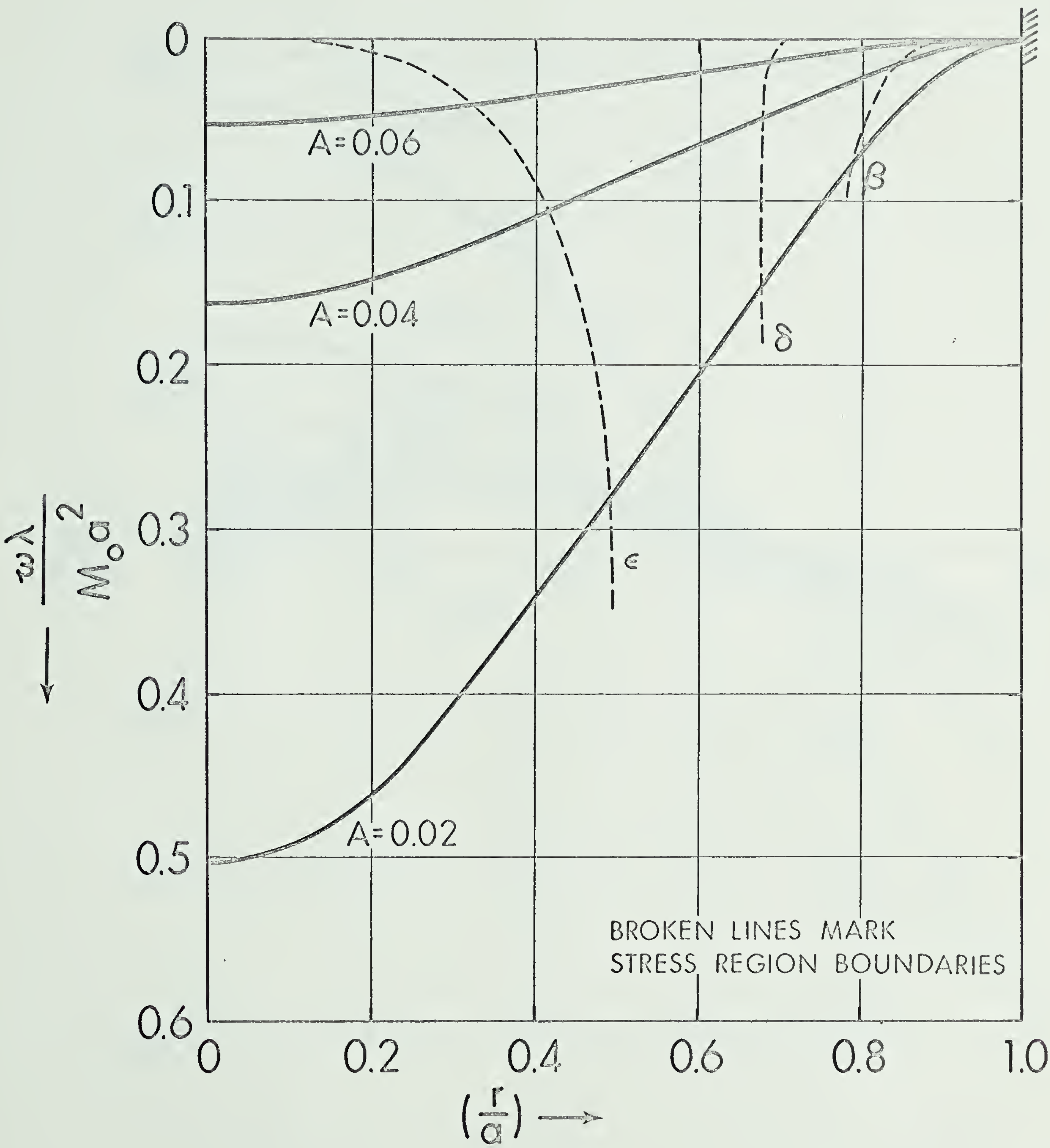

FIG. 3.4

VISCO-PLASTIC PLATE

DEFLECTION RATE vs. RADIUS

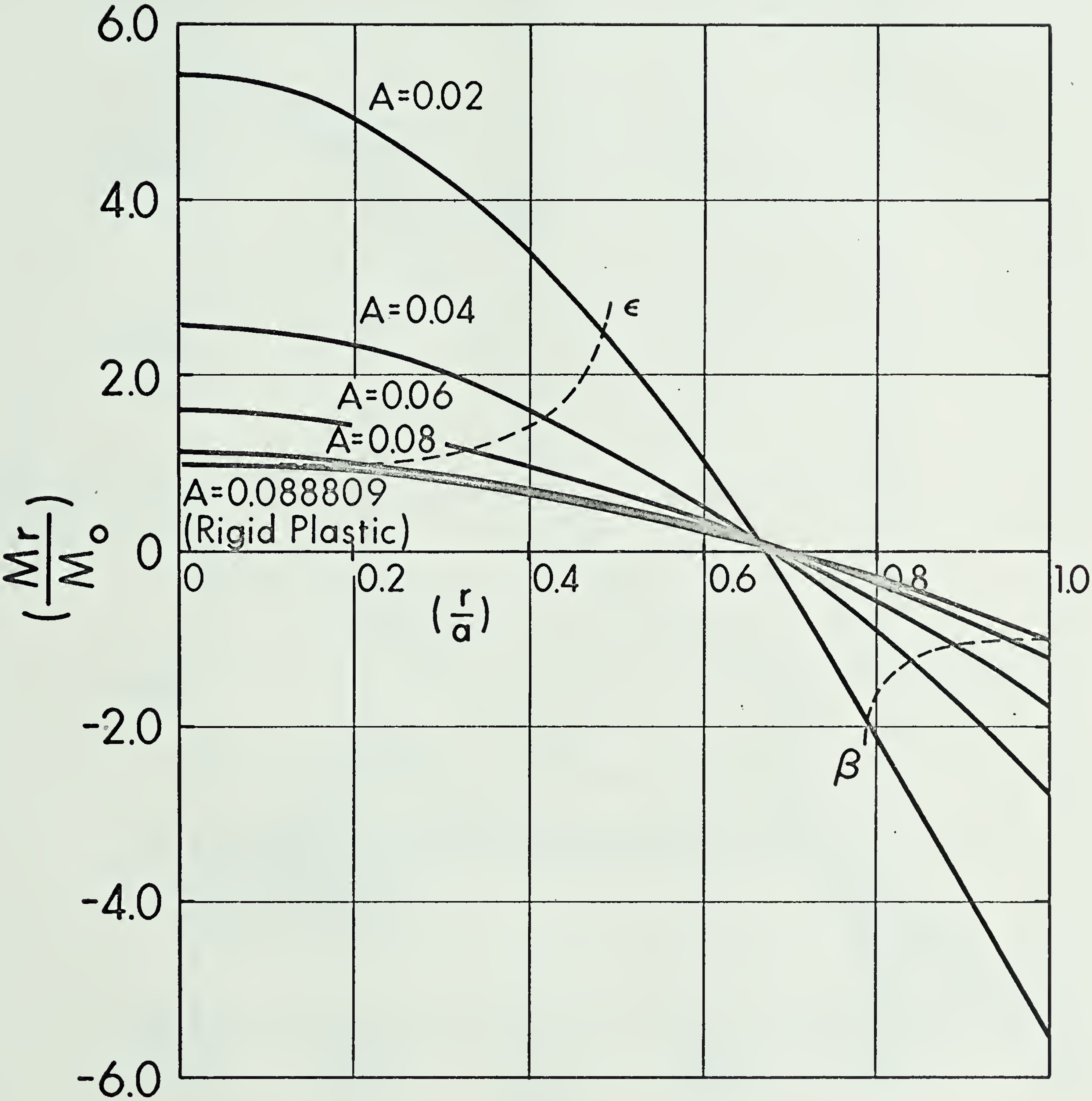

FIG. 3.5

VISCO-PLASTIC PLATE

BENDING MOMENT M_r vs. RADIUS

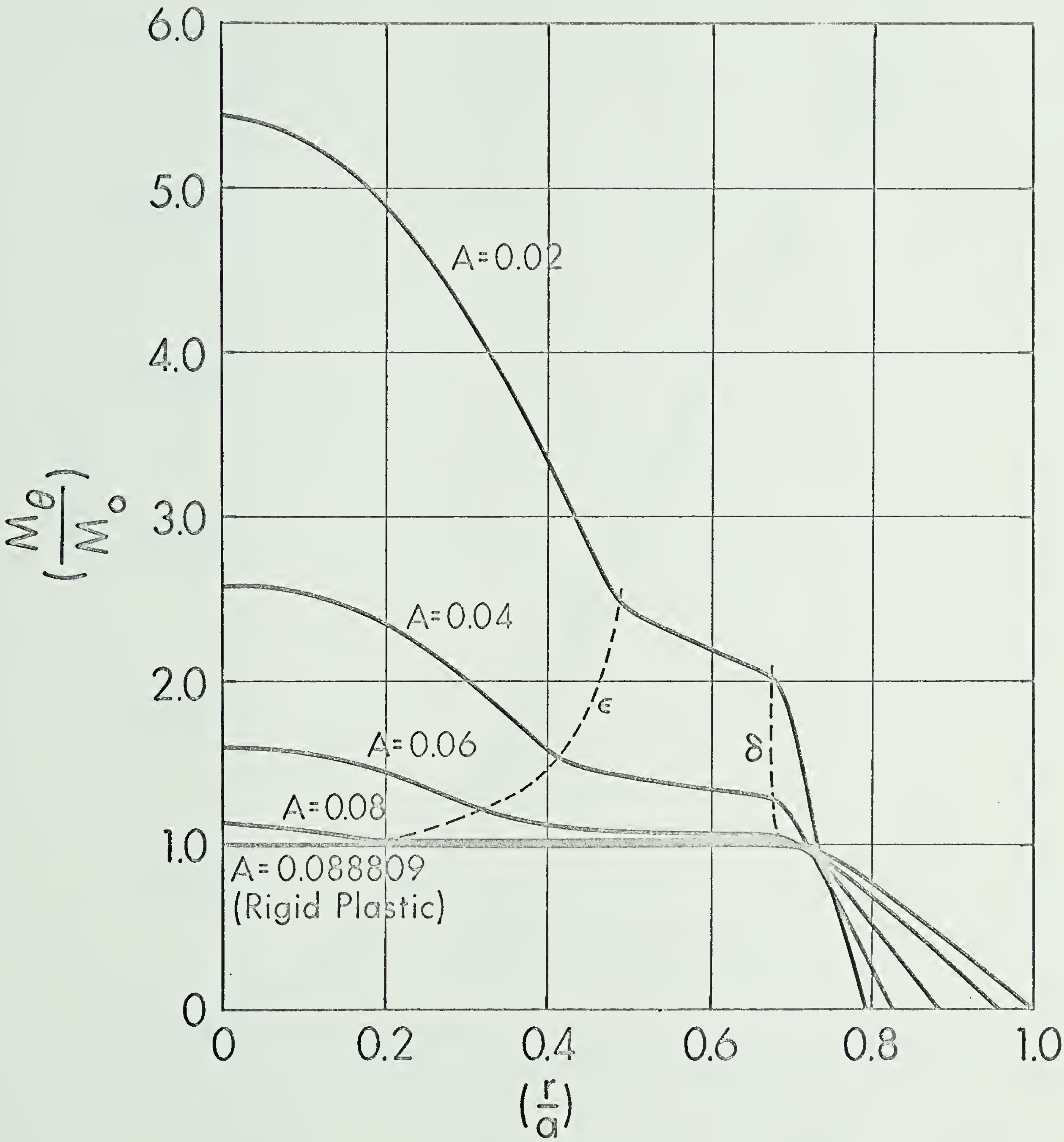

FIG. 3.6

VISCO-PLASTIC PLATE

BENDING MOMENT M_θ vs. RADIUS

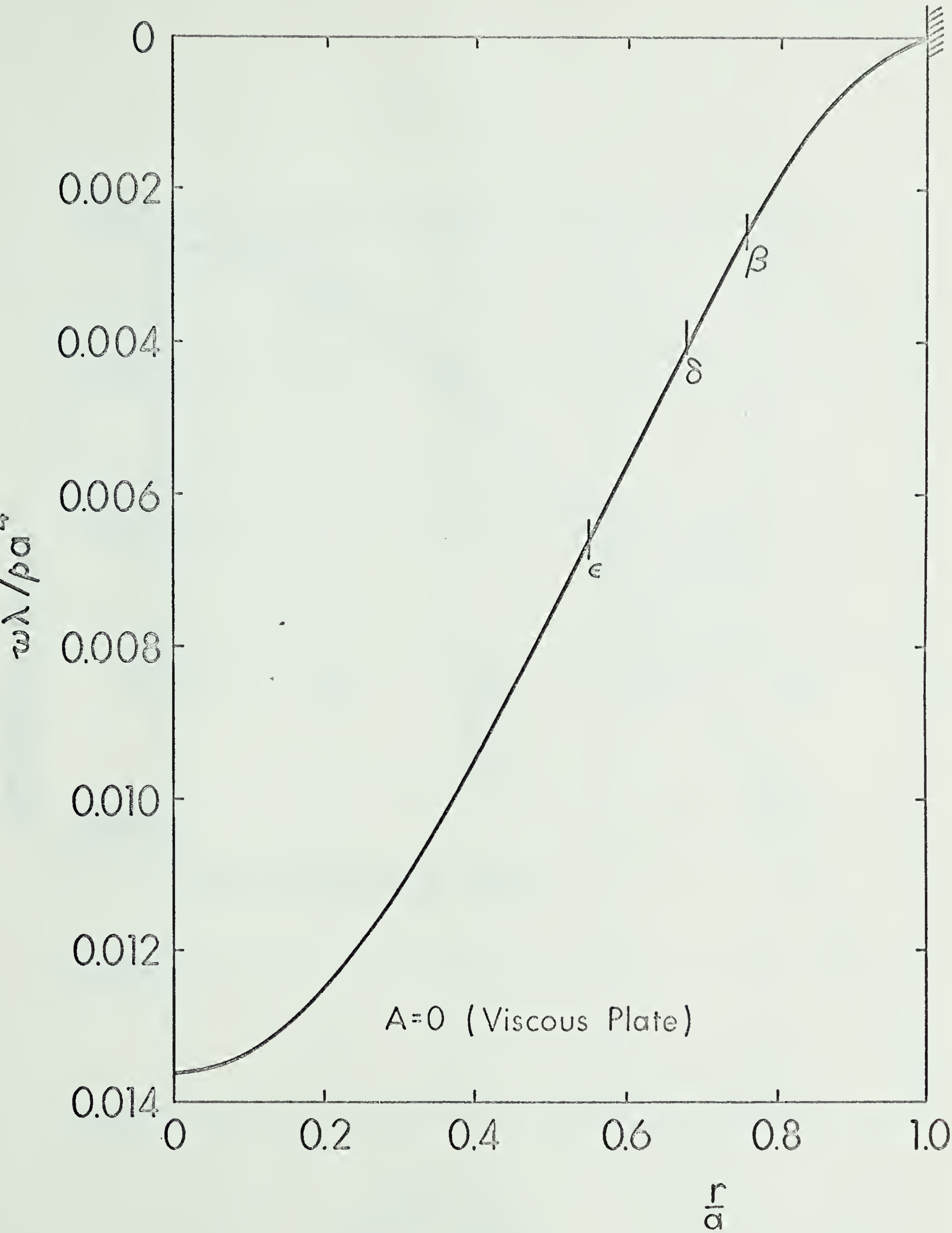

FIG. 3.7

VISCOUS PLATE

DEFLECTION RATE vs. RADIUS

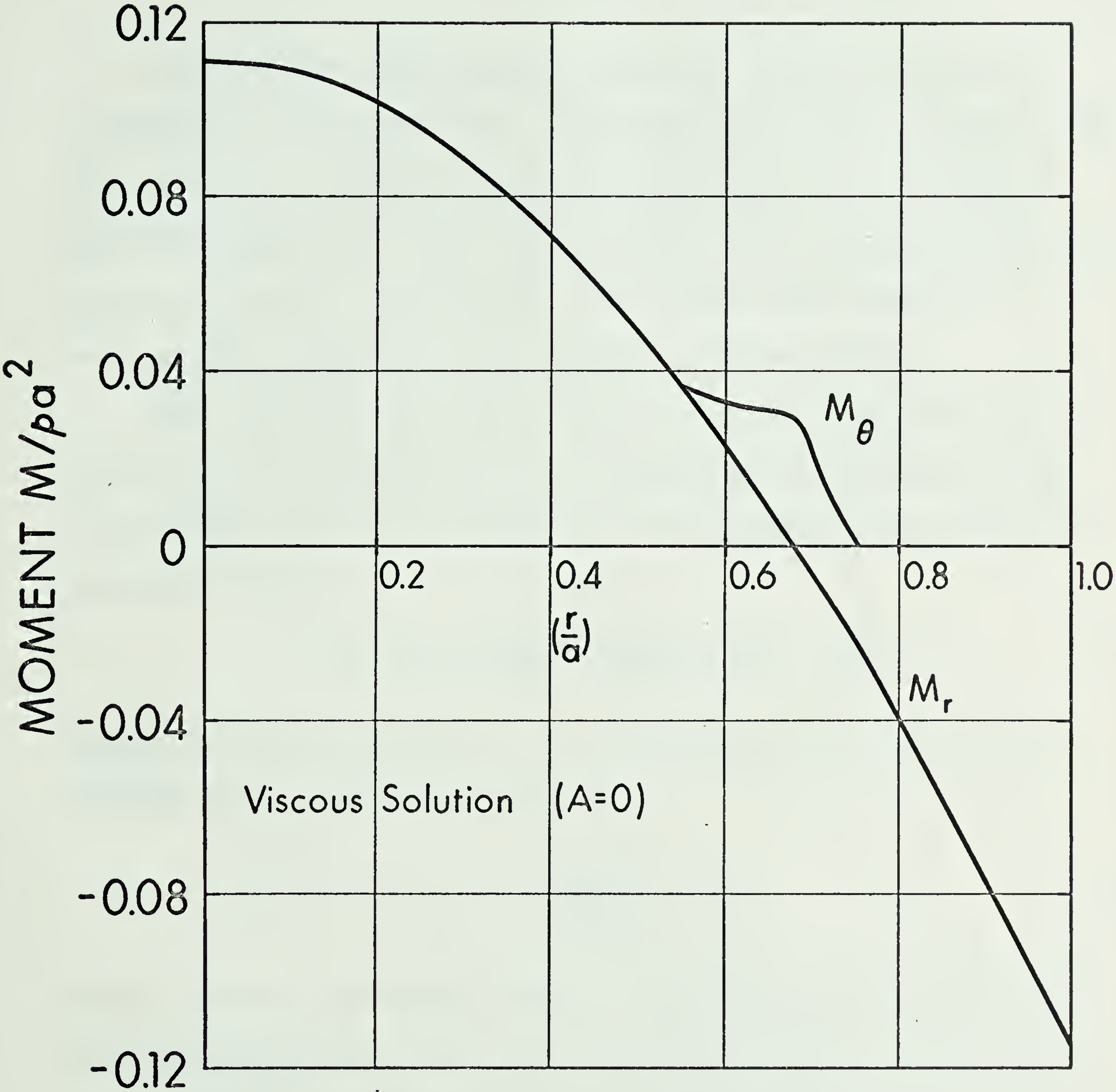

FIG. 3.8

VISCOUS PLATE

BENDING MOMENT vs. RADIUS

CHAPTER IV

CREEP PROBLEM

A different constitutive relation is now applied to a sandwich plate otherwise identical with that in Chapter III. The method of solution which is used was first introduced by Venkatraman and Hodge [16], however the complementary function was not introduced. The method was applied to both simply supported and clamped circular plates.

The constitutive equation for the material now considered can be obtained from the complementary function which is assumed equal to a function of the generalized stress intensity M where

$$M = \max\left[\,|M_r|\,,\ |M_\theta|\,,\ |M_r - M_\theta|\,\right] \tag{4.1}$$

times a function of time. If stationary creep only is considered then E_c may be chosen as

$$E_c = \frac{(M)^{n+1}}{(n+1)\mu^n}\ G(t) \tag{4.2}$$

where μ and n are creep constants depending on the material and the temperature and μ has the dimensions of moment per unit length. The constitutive relation found from (4.2) represents time hardening creep. The time hardening effect is introduced by the term G(t). This surface of constant E_c is a hexagonal prism in principal stress space.

The principal curvature rates are

$$\kappa_r = \frac{\partial E_c}{\partial M_r} \qquad \text{and} \qquad \kappa_\theta = \frac{\partial E_c}{\partial M_\theta} \ . \tag{4.3}$$

When n approaches infinity the constitutive relation defined by (4.2) and (4.3) approaches that for the Tresca rigid-perfectly plastic plate problem with a yield moment equal to μ. When n is one this relation is essentially equivalent to that given by (1.23).

The complete flow rule derived from (4.2) using (4.3) and the condition that the rate of energy dissipation per unit area is constant on the E_c surface is given in Table V.

TABLE V

FLOW RULE FROM EQUATION (4.2)

Stress Regime	Complementary Function	Flow Rule	
COB	$\frac{M_\theta^{n+1}}{(n+1)\mu^n} G(t)$	$\kappa_r = 0$	
		$\kappa_\theta = \left(\frac{M_\theta}{\mu}\right)^n G(t)$	
OC	$\frac{M_\theta^{n+1}}{(n+1)\mu^n} G(t)$	$\kappa_r = -s\left(\frac{M_\theta - M_r}{\mu}\right)^n G(t)$	$0 \leqslant s \leqslant 1$
		$\kappa_\theta = (1-s)\left(\frac{M_\theta}{\mu}\right)^n G(t) + s\left(\frac{M_\theta - M_r}{\mu}\right)^n G(t)$	

Stress Regime	Complementary Function	Flow Rule
COD	$\dfrac{(M_\theta - M_r)^{n+1} G(t)}{(n+1)\ \mu^n}$	$\kappa_r = -\left(\dfrac{M_\theta - M_r}{\mu}\right)^n G(t)$ $\kappa_\theta = +\left(\dfrac{M_\theta - M_r}{\mu}\right)^n G(t)$
OD	$\dfrac{(-M_r)^{n+1}}{(n+1)\ \mu^n} G(t)$	$\kappa_r = -t\left(\dfrac{-M_r}{\mu}\right)^n G(t) + (t-1)\left(\dfrac{M_\theta - M_r}{\mu}\right)^n G(t)$ $\kappa_\theta = (1-t)\left(\dfrac{M_\theta - M_r}{\mu}\right)^n G(t) \qquad 0 \leqslant t \leqslant 1$
DOE	$\dfrac{(-M_r)^{n+1}}{(n+1)\ \mu^n} G(t)$	$\kappa_r = -\left(\dfrac{-M_r}{\mu}\right)^n G(t)$ $\kappa_\theta = 0$
OE	$\dfrac{(-M_\theta)^{n+1}}{(n+1)\ \mu^n} G(t)$	$\kappa_r = -q\left(\dfrac{-M_\theta}{\mu}\right)^n G(t)$ $\kappa_\theta = (q-1)\left(\dfrac{-M_\theta}{\mu}\right)^n G(t) \qquad 0 \leqslant q \leqslant 1$
EOF	$\dfrac{(-M_\theta)^{n+1}}{(n+1)\ \mu^n} G(t)$	$\kappa_r = 0$ $\kappa_\theta = -\left(\dfrac{-M_\theta}{\mu}\right)^n G(t)$
OF	$\dfrac{(-M_\theta)^{n+1}}{(n+1)\ \mu^n} G(t)$	$\kappa_r = m\left(\dfrac{M_r - M_\theta}{\mu}\right)^n G(t) \qquad 0 \leqslant m \leqslant 1$ $\kappa_\theta = (m-1)\left(\dfrac{-M_\theta}{\mu}\right)^n G(t) - m\left(\dfrac{M_r - M_\theta}{\mu}\right)^n G(t)$

Stress Regime	Complementary Function	Flow Rule
FOA	$\dfrac{(M_r - M_\theta)^{n+1}}{(n+1)\ \mu^n}\ G(t)$	$\kappa_r = \left(\dfrac{M_r - M_\theta}{\mu}\right)^n G(t)$
		$\kappa_\theta = -\left(\dfrac{M_r - M_\theta}{\mu}\right)^n G(t)$
OA	$\dfrac{(M_r)^{n+1}}{(n+1)\ \mu^n}\ G(t)$	$\kappa_r = g\left(\dfrac{M_r - M_\theta}{\mu}\right)^n G(t) + (1-g)\left(\dfrac{M_r}{\mu}\right)^n G(t)$
		$\kappa_\theta = -g\left(\dfrac{M_r - M_\theta}{\mu}\right)^n G(t) \qquad 0 \leqslant g \leqslant 1$
AB	$\dfrac{(M_r)^{n+1}}{(n+1)\ \mu^n}\ G(t)$	$\kappa_r = \left(\dfrac{M_r}{\mu}\right)^n G(t)$
		$\kappa_\theta = 0$
OB	$\dfrac{(M_r)^{n+1}}{(n+1)\ \mu^n}\ G(t)$	$\kappa_r = f\left(\dfrac{M_r}{\mu}\right)^n G(t)$
		$\kappa_\theta = (1-f)\left(\dfrac{M_\theta}{\mu}\right)^n G(t) \qquad 0 \leqslant f \leqslant 1$

An a priori assumption is made for the stress profile. The stress profile of Table III is assumed.

The boundary conditions for the problem are

$$\frac{dw}{dr}(0) = 0 \tag{4.4}$$

$$\frac{dw}{dr}(a) = 0 \tag{4.5}$$

$$w(a) = 0 \tag{4.6}$$

As in the previous problem the bending moments must satisfy the equilibrium equation (3.5) and the curvature rates must be derivable from the velocity using equations (3.1).

In region OB the stress intensity M is equal to M_r and M_θ. Therefore addition of the curvature rates gives

$$\kappa_r + \kappa_\theta = \left(\frac{M_\theta}{\mu}\right)^n G(t) . \qquad 0 \leqslant r \leqslant e \tag{4.7}$$

The equilibrium equation in this region, $0 \leqslant r \leqslant e$, becomes

$$r \frac{dM_r}{dr} + \frac{pr^2}{2} = 0 .$$

Integration of this gives

$$M_r = -\frac{pr^2}{4} + c_1 = M_\theta . \tag{4.8}$$

From equations (3.1), (4.7), and (4.8)

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) = \left(\frac{M_r}{\mu}\right)^n G(t) = G(t)\left(\frac{\frac{-pr^2}{4} + c_1}{\mu}\right)^n .$$

Integration and the boundary condition (4.4) gives

$$\frac{dw}{dr} = \frac{2G}{p\mu^n(n+1)}\left[\left(-\frac{pr^2}{4} + c_1\right)^{n+1} - c_1^{\,n+1}\right]\left(\frac{1}{r}\right)$$

from which

$$w = \frac{2G}{p\mu^n(n+1)}\int\left[\left(-\frac{pr^2}{4} + c_1\right)^{n+1} - c_1^{\,n+1}\right]\frac{dr}{r} \tag{4.9}$$

$$0 \leqslant r \leqslant e$$

In the second region of the plate, $e \leqslant r \leqslant d$, the curvature rate κ_r is zero and therefore

$$w = -c_2 r + c_3 . \qquad e \leqslant r \leqslant d \tag{4.10}$$

From Table V and equation (3.1)

$$\left(\frac{M_\theta}{\mu}\right)^n G = \frac{c_2}{r}$$

which gives

$$M_\theta = \mu \left(\frac{c_2}{Gr}\right)^{1/n} . \tag{4.11}$$

Substituting this into the equilibrium equation and integrating results in

$$M_r = \frac{n\mu}{(n-1)} \left(\frac{c_2}{Gr}\right)^{1/n} - \frac{pr^2}{6} + \frac{c_4}{r} . \qquad e \leqslant r \leqslant d$$

This expression is valid for n not equal to one. When n is equal to one

$$M_r = \frac{\mu c_2}{Gr} \ln r - \frac{pr^2}{6} + \frac{c_4}{r} .$$

This latter expression will not be used thereby restricting n to be other than one. The particular problem when n does equal one is equivalent to the visco-plastic plate problem solved in Chapter III, with the yield moment equal to zero.

In the third region of the plate, $d \leqslant r \leqslant b$, the curvature rates are equal in magnitude and opposite in sign so that addition gives

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} = 0 \tag{4.12}$$

from which

$$w = -c_5 \ln r + c_6 \, . \qquad d \leqslant r \leqslant b \tag{4.13}$$

Use of (4.13) in the expression for κ_θ in this region from Table V gives

$$\frac{c_5}{r^2} = \left(\frac{M_\theta - M_r}{\mu}\right)^n G$$

or

$$M_\theta - M_r = \mu \left(\frac{c_5}{Gr^2}\right)^{1/n} \tag{4.14}$$

Substitution of this latter expression into the equilibrium equation gives

$$r\frac{dM_r}{dr} - \mu \left(\frac{c_5}{Gr^2}\right)^{1/n} + \frac{pr^2}{2} = 0 \tag{4.15}$$

from which

$$M_r = \frac{-n\mu}{2}\left(\frac{c_5}{Gr^2}\right)^{1/n} - \frac{pr^2}{4} + c_7 \tag{4.16}$$

and

$$M_\theta = \mu\left(\frac{2-n}{2}\right)\left(\frac{c_5}{Gr^2}\right)^{1/n} - \frac{pr^2}{4} + c_7 \tag{4.17}$$

$$\left.\right\} d \leqslant r \leqslant b$$

In the outermost region of the plate, $b \leqslant r \leqslant a$, the moment M_θ is zero. Thus integration of the equilibrium equation gives

$$M_r = -\frac{pr^2}{6} + \frac{c_8}{r} \, . \qquad b \leqslant r \leqslant a \tag{4.18}$$

From equation (4.18) and the flow rule for this region

$$-\frac{d^2w}{dr^2} = -G\left(-\frac{M_r}{\mu}\right)^n = -\frac{G}{\mu^n}\left(\frac{pr^2}{6} - \frac{c_8}{r}\right)^n \tag{4.19}$$

Integration of this and use of boundary condition (4.5) gives

$$\frac{dw}{dr} = -\frac{G}{\mu^n}\int_r^a \left[\frac{p\zeta^2}{6} - \frac{c_8}{\zeta}\right]^n d\zeta \tag{4.20}$$

Integrating once more and use of boundary condition (4.6) results in

$$w = \frac{G}{\mu^n}\int_r^a\int_\xi^a \left(\frac{p\zeta^2}{6} - \frac{c_8}{\zeta}\right)^n d\zeta d\xi \;. \qquad b \leqslant r \leqslant a \tag{4.21}$$

Equilibrium requires that M_r and M_θ be continuous. Also the viscosity effect requires that $\frac{dw}{dr}$ be continuous, and since the plate does not fracture, w is continuous. Therefore these continuity conditions applied to the three boundaries of the stress regions result in a total of thirteen equations.

Continuity of w, $\frac{dw}{dr}$, M_r, and M_θ respectively at r = e gives

$$\frac{2G}{p\mu^n(n+1)}\left\{\int\left[\left(-\frac{pr^2}{4} + c_1\right)^{n+1} - c_1^{\,n+1}\right]\frac{dr}{r}\right\}_{r=e} = -c_2e + c_3 \tag{4.22}$$

$$\frac{2G}{p\mu^n(n+1)}\left[\left(-\frac{pe^2}{4} + c_1\right)^{n+1} - c_1^{\,n+1}\right]\frac{1}{e} = -c_2 \tag{4.23}$$

$$-\frac{pe^2}{4} + c_1 = \frac{n\mu}{n-1}\left(\frac{c_2}{Ge}\right)^{1/n} - \frac{pe^2}{6} + \frac{c_4}{e} \tag{4.24}$$

$$-\frac{pe^2}{4} + c_1 = \mu\left(\frac{c_2}{Ge}\right)^{1/n} . \tag{4.25}$$

At r = d these continuity conditions and the condition $M_r(d) = 0$ gives

$$- c_2 d + c_3 = -c_5 \ln d + c_6 \tag{4.26}$$

$$- c_2 = -c_5/d \tag{4.27}$$

$$\frac{n\mu}{n-1}\left(\frac{c_2}{Gd}\right)^{1/n} - \frac{pd^2}{6} + \frac{c_4}{d} = 0 \tag{4.28}$$

$$- \frac{n\mu}{2}\left(\frac{c_5}{Gd^2}\right)^{1/n} - \frac{pd^2}{4} + c_7 = 0 \tag{4.29}$$

$$\mu\left(\frac{c_2}{Gd}\right)^{1/n} = \mu\left(\frac{2-n}{2}\right)\left(\frac{c_5}{Gd^2}\right)^{1/n} - \frac{pd^2}{4} + c_7 \, . \tag{4.30}$$

At $r = b$ the following equations are obtained.

$$-c_5 \ln b + c_6 = \frac{G}{\mu^n}\int_b^a \int_\xi^a \left[\frac{p\zeta^2}{6} - \frac{c_8}{\zeta}\right]^n d\zeta d\xi \tag{4.31}$$

$$-c_5/b = - \frac{G}{\mu^n}\int_b^a \left[\frac{p\zeta^2}{6} - \frac{c_8}{\zeta}\right]^n d\zeta \tag{4.32}$$

$$- \frac{n\mu}{2}\left(\frac{c_5}{Gb^2}\right)^{1/n} - \frac{pb^2}{4} + c_7 = - \frac{pb^2}{6} + \frac{c_8}{b} \tag{4.33}$$

$$\mu\left(\frac{2-n}{2}\right)\left(\frac{c_5}{Gb^2}\right)^{1/n} - \frac{pb^2}{4} + c_7 = 0 \tag{4.34}$$

The unknowns are e, d, b, c_1 to c_8, and one other constant of integration from equation (4.9). There is a total of twelve unknowns and what appears to be thirteen simultaneous independent equations. All thirteen equations are not independent however since equation (4.29) is a combination of equations (4.27) and (4.30).

Using equation (4.22) to eliminate the constant of

integration in equation (4.9) gives

$$w = \frac{2G}{p\mu^n(n+1)} \int_r^e \left[c_1^{\,n+1} - \left(- \frac{p\zeta^2}{4} + c_1 \right)^{n+1} \right] \frac{d\zeta}{\zeta}$$

$$- c_2 e + c_3 \ . \quad 0 \leqslant r \leqslant e \qquad (4.35)$$

Solving equation (4.28) for c_4 gives

$$c_4 = \frac{pd^3}{6} - \frac{n\mu d}{n-1} \left(\frac{c_2}{Gd} \right)^{1/n} . \qquad (4.36)$$

Subtracting equation (4.25) from equation (4.24) results in

$$\frac{p}{6e} \left(d^3 - e^3 \right) + \frac{\mu}{n-1} \left(\frac{c_2}{G} \right)^{1/n} \left[\left(\frac{1}{e} \right)^{1/n} - \frac{nd^{\frac{n-1}{n}}}{e} \right] = 0 \ . \qquad (4.37)$$

Solving equations (4.36) and (4.37) for c_2 and c_4 yields

$$c_2 = G \left[\frac{p(e^3-d^3)(n-1)}{6\mu\left(e^{\frac{n-1}{n}} - nd^{\frac{n-1}{n}} \right)} \right]^n$$

and

$$c_4 = \frac{pd^3}{6} - \left(\frac{n}{n-1} \right) d^{\frac{n-1}{n}} \left[\frac{p(e^3-d^3)(n-1)}{6\left(e^{\frac{n-1}{n}} - nd^{\frac{n-1}{n}} \right)} \right] .$$

Substituting c_2 in equation (4.25) results in

$$c_1 = \frac{pe^2}{4} + \frac{p(e^3-d^3)(n-1)}{6e^{1/n}\left(e^{\frac{n-1}{n}} - nd^{\frac{n-1}{n}} \right)}$$

From equation (4.27)

$$c_5 = Gd\left[\frac{p(e^3-d^3)(n-1)}{6\mu\left(e^{\frac{n-1}{n}} - nd^{\frac{n-1}{n}}\right)}\right]^n .$$

The above and equation (4.29) or equations (4.27) and (4.30) gives

$$c_7 = \frac{n}{12}\left[\frac{p(e^3-d^3)(n-1)}{d^{1/n}\left(e^{\frac{n-1}{n}} - nd^{\frac{n-1}{n}}\right)}\right] + \frac{pd^2}{4} .$$

Substitution of the expressions for c_5 and c_7 into equation (4.33) yields

$$c_8 = p\left(-\frac{b^3}{12} + \frac{bd^2}{4}\right) + \frac{n(n-1)p(e^3-d^3)}{12\left(e^{\frac{n-1}{n}} - nd^{\frac{n-1}{n}}\right)}\left(\frac{b}{d^{1/n}} - b^{\frac{n-2}{n}} d^{1/n}\right) .$$

From equation (4.31) the following is obtained

$$c_6 = Gd \ln b \left[\frac{p(e^3-d^3)(n-1)}{6\mu\left(e^{\frac{n-1}{n}} - nd^{\frac{n-1}{n}}\right)}\right]^n + \frac{G}{\mu^n}\int_b^a\int_\xi^a\left[\frac{p}{12}\left(2\zeta^2 + \frac{b^3}{\zeta}\right.\right.$$

$$\left.\left. - \frac{3bd^2}{\zeta}\right) - \frac{n(n-1)p(e^3-d^3)(b/d^{1/n} - b^{\frac{n-2}{n}} d^{1/n})}{12\left(e^{\frac{n-1}{n}} - nd^{\frac{n-1}{n}}\right)}\right]^n d\zeta d\xi .$$

Equation (4.26) gives

$$c_3 = Gd(1+\ln\frac{b}{d})\left[\frac{p(e^3-d^3)(n-1)}{6\mu\left(e^{\frac{n-1}{n}} - nd^{\frac{n-1}{n}}\right)}\right]^n + \frac{G}{\mu^n}\int_b^a\int_\xi^a\left[\frac{p}{12}\left(2\zeta^2 + \frac{b^3}{\zeta}\right.\right.$$

$$\left.\left. - \frac{3bd^2}{\zeta}\right) - \frac{n(n-1)p(e^3-d^3)(b/d^{1/n} - b^{\frac{n-2}{n}} d^{1/n})}{12\left(e^{\frac{n-1}{n}} - nd^{\frac{n-1}{n}}\right)}\right]^n d\zeta d\xi .$$

The three remaining equations may now be used to find the values of e, d, and b. The following non-dimensional parameters are used.

$$\epsilon = \frac{e}{a}, \qquad \delta = \frac{d}{a}, \qquad \beta = \frac{b}{a}, \qquad \rho = \frac{r}{a}$$

Substitution of the expressions obtained for the constants c_1 to c_8 into equations (4.23), (4.32), and (4.34) gives the following three equations respectively,

$$\frac{2}{(n+1)\epsilon}\left[\left\{\frac{(\epsilon^3-\delta^3)(n-1)}{6\epsilon^{1/n}\left(\epsilon^{\frac{n-1}{n}} - n\delta^{\frac{n-1}{n}}\right)}\right\}^{n+1} - \left\{\frac{\epsilon^2}{4} + \frac{(\epsilon^3-\delta^3)(n-1)}{6\epsilon^{1/n}\left(\epsilon^{\frac{n-1}{n}} - n\delta^{\frac{n-1}{n}}\right)}\right\}^{n+1}\right]$$

$$+ \left[\frac{(\epsilon^3-\delta^3)(n-1)}{6\left(\epsilon^{\frac{n-1}{n}} - n\delta^{\frac{n-1}{n}}\right)}\right]^n = 0 \qquad (4.38)$$

$$-\frac{\delta}{\beta}\left[\frac{2(\epsilon^3-\delta^3)(n-1)}{\left(\epsilon^{\frac{n-1}{n}} - n\delta^{\frac{n-1}{n}}\right)}\right]^n + \int_\beta^1 \left\{ 2\zeta^2 + \beta^3/\zeta - 3\beta\delta^2/\zeta \right.$$

$$\left. - \frac{n(n-1)(\epsilon^3-\delta^3)}{\left(\epsilon^{\frac{n-1}{n}} - n\delta^{\frac{n-1}{n}}\right)}\left[\frac{\beta}{\delta^{1/n}} - \beta^{\frac{n-2}{n}}\delta^{1/n}\right]\right\}^n d\zeta = 0 \qquad (4.39)$$

$$\left[\frac{(\epsilon^3-\delta^3)(n-1)}{\left(\epsilon^{\frac{n-1}{n}} - n\delta^{\frac{n-1}{n}}\right)}\right]\left[(2-n)\left(\frac{\delta}{\beta^2}\right)^{1/n} + \frac{n}{\delta^{1/n}}\right] + 3(\delta^2-\beta^2) = 0. \qquad (4.40)$$

These three equations must be solved simultaneously for each value of n. Solutions have been found using a computer. The integration in equation (4.39) was done numerically using a 24 point Gauss Quadrature routine [11]. The roots of the three equations were found using Newton's Method. The results are shown in Table VI.

TABLE VI

STRESS REGION BOUNDARIES FOR CREEP PROBLEM

n	ϵ	δ	β
2	0.43009	0.67414	0.80074
4	0.32217	0.68378	0.85900
6	0.26745	0.69077	0.88934
8	0.23332	0.69581	0.90820
10	0.20951	0.69959	0.92116
12	0.19170	0.70254	0.93066

The values of ϵ, δ, and β for n = 1 are the same as those for the visco-plastic problem when the yield stress goes to zero.

As n approaches infinity the constitutive relation from equations (4.2) and (4.3) approaches that for a Tresca rigid-perfectly plastic plate. The results in Table V show that as n becomes large the values of ϵ, δ, and β approach rather slowly those for the Tresca rigid-perfectly plastic

plate problem. The values for the latter problem being $\epsilon = 0$, $\delta = 0.730$, and $\beta = 1.0$.

Substitution of the values of c_1 to c_8 into the expressions for w, M_r, and M_θ for each region of the plate results in a complete solution since ϵ, δ, and β can be determined for any value of n from equations (4.38), (4.39), and (4.40).

The velocity solution is

$$w = Ga^2 \left(\frac{pa^2}{6\mu}\right)^n \left(\frac{1}{3} \int_\rho^\epsilon \left\{ \left[\frac{3\epsilon^2}{2} + \frac{(\epsilon^3-\delta^3)(n-1)}{\epsilon^{1/n}\left(\epsilon^{\frac{n-1}{n}} - n\delta^{\frac{n-1}{n}}\right)}\right]^{n+1} - \left[\frac{3}{2}(\epsilon^2-\zeta^2) + \frac{(\epsilon^3-\delta^3)(n-1)}{\epsilon^{1/n}\left(\epsilon^{\frac{n-1}{n}} - n\delta^{\frac{n-1}{n}}\right)}\right]^{n+1} \right\} \frac{d\zeta}{\zeta} + \left[\delta + \delta \ln \beta/\delta - \epsilon\right] \left[\frac{(\epsilon^3-\delta^3)(n-1)}{\left(\epsilon^{\frac{n-1}{n}} - n\delta^{\frac{n-1}{n}}\right)}\right]^n + B\right) \qquad 0 \leqslant \rho \leqslant \epsilon$$

$$w = Ga^2 \left(\frac{pa^2}{6\mu}\right)^n \left((\delta + \delta \ln \beta/\delta - \rho)\left[\frac{(\epsilon^3-\delta^3)(n-1)}{\left(\epsilon^{\frac{n-1}{n}} - n\delta^{\frac{n-1}{n}}\right)}\right]^n + B\right) \qquad \epsilon \leqslant \rho \leqslant \delta$$

$$w = Ga^2 \left(\frac{pa^2}{6\mu}\right)^n \left((-\delta \ln \rho/\beta)\left[\frac{(\epsilon^3-\delta^3)(n-1)}{\left(\epsilon^{\frac{n-1}{n}} - n\delta^{\frac{n-1}{n}}\right)}\right]^n + B\right) \qquad \delta \leqslant \rho \leqslant \beta$$

$$w = Ga^2\left(\frac{pa^2}{6\mu}\right)^n \int_\rho^1 \int_\xi^1 \left[\left(\zeta^2 + \frac{\beta^3}{2\zeta} - \frac{3\beta\delta^2}{2\zeta}\right) - \frac{n(n-1)(\epsilon^3-\delta^3)(\beta/\delta^{1/n} - \beta^{\frac{n-2}{n}}\delta^{1/n})}{2\left(\epsilon^{\frac{n-1}{n}} - n\delta^{\frac{n-1}{n}}\right)}\right]^n d\zeta d\xi \qquad \beta \leqslant \rho \leqslant 1$$

Where

$$B = \int_\beta^1 \int_\xi^1 \left[\left(\zeta^2 + \frac{\beta^3}{2\zeta} - \frac{3\beta\delta^2}{2\zeta}\right) - \frac{n(n-1)(\epsilon^3-\delta^3)(\beta/\delta^{1/n} - \beta^{\frac{n-2}{n}}\delta^{1/n})}{2\left(\epsilon^{\frac{n-1}{n}} - n\delta^{\frac{n-1}{n}}\right)}\right]^n d\zeta d\xi \ .$$

The stress solution is

$$M_r = M_\theta = \left(\frac{pa^2}{6}\right)\left[\frac{3}{2}(\epsilon^2-\rho^2) + \frac{(\epsilon^3-\delta^3)(n-1)}{\epsilon^{1/n}\left(\epsilon^{\frac{n-1}{n}} - n\delta^{\frac{n-1}{n}}\right)}\right] \qquad 0 \leqslant \rho \leqslant \epsilon$$

$$M_r = \frac{pa^2}{6}\left[\left(\frac{\delta^3-\rho^3}{\rho}\right) + \left\{\left(\frac{1}{\rho}\right)^{1/n} - \frac{\delta}{\rho}\left(\frac{1}{\delta}\right)^{1/n}\right\}\left\{\frac{(\epsilon^3-\delta^3)n}{\epsilon^{\frac{n-1}{n}} - n\delta^{\frac{n-1}{n}}}\right\}\right]$$

$$\epsilon \leqslant \rho \leqslant \delta$$

$$M_\theta = \frac{pa^2}{6}\left[\frac{(n-1)(\epsilon^3-\delta^3)}{\left(\epsilon^{\frac{n-1}{n}} - n\delta^{\frac{n-1}{n}}\right)}\left(\frac{1}{\rho}\right)^{1/n}\right]$$

$$M_r = \frac{pa^2}{6}\left[\frac{3}{2}(\delta^2-\rho^2) + \frac{n}{2}\left(\frac{1}{\delta^{1/n}} - \frac{\delta^{1/n}}{\rho^{2/n}}\right)\left(\frac{(\epsilon^3-\delta^3)(n-1)}{\epsilon^{\frac{n-1}{n}} - n\delta^{\frac{n-1}{n}}}\right)\right]$$

$$\delta \leqslant \rho \leqslant \beta$$

$$M_\theta = \frac{pa^2}{6}\left[\frac{3}{2}(\delta^2-\rho^2) + \left\{\frac{\delta^{1/n}}{\rho^{2/n}} + \frac{n}{2}\left(\frac{1}{\delta^{1/n}} - \frac{\delta^{1/n}}{\rho^{2/n}}\right)\right\}\left\{\frac{(\epsilon^3-\delta^3)(n-1)}{\left(\epsilon^{\frac{n-1}{n}} - n\delta^{\frac{n-1}{n}}\right)}\right\}\right]$$

$$M_r = \frac{pa^2}{6}\left[\left(-\rho^2 - \frac{\beta^3}{2\rho} + \frac{3\beta\delta^2}{\rho}\right) + \frac{n(n-1)(\epsilon^3-\delta^3)}{2\rho\left(\epsilon^{\frac{n-1}{n}} - n\delta^{\frac{n-1}{n}}\right)}\left(\frac{\beta}{\delta^{1/n}} - \beta^{\frac{n-2}{n}}\delta^{1/n}\right)\right]$$

$$\beta \leqslant \rho \leqslant 1$$

$$M_\theta = 0 \ .$$

Numerical results have been computed for w, M_r, and M_θ using the above expressions. These values have been plotted in Figures (4.1), (4.2) and (4.3).

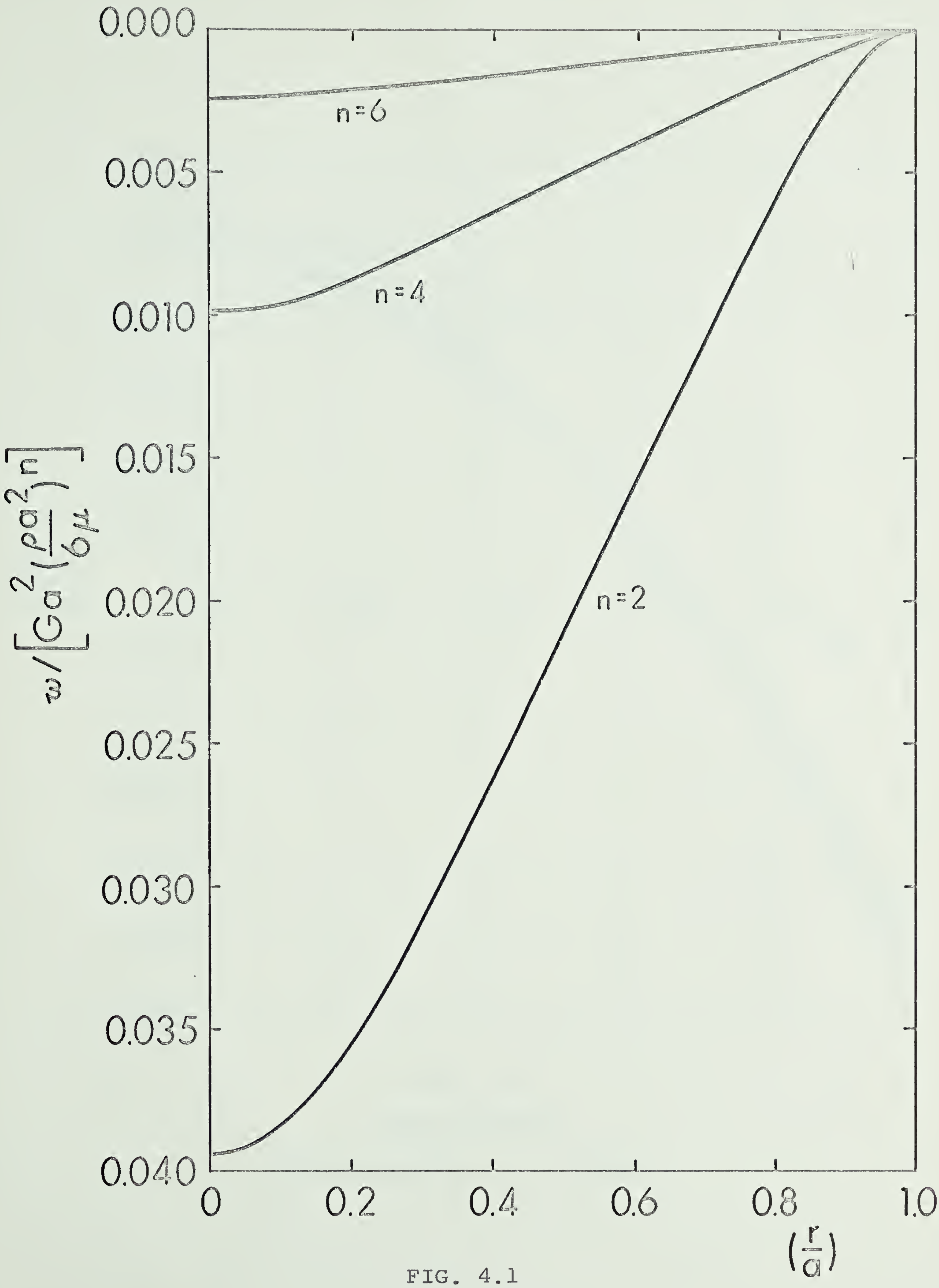

FIG. 4.1
CREEP PROBLEM
DEFLECTION RATE vs. RADIUS

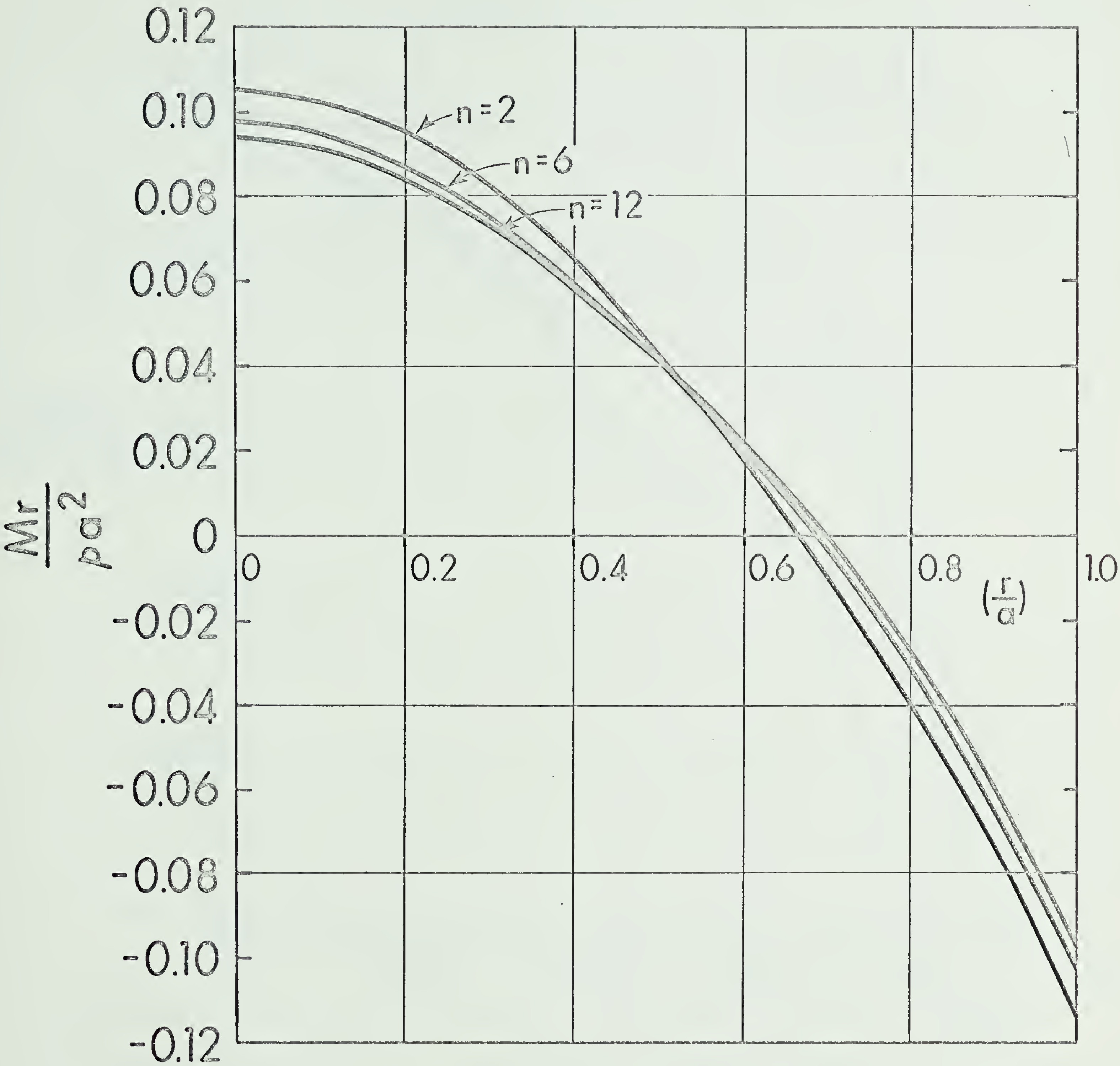

FIG. 4.2

CREEP PROBLEM

BENDING MOMENT M_r vs. RADIUS

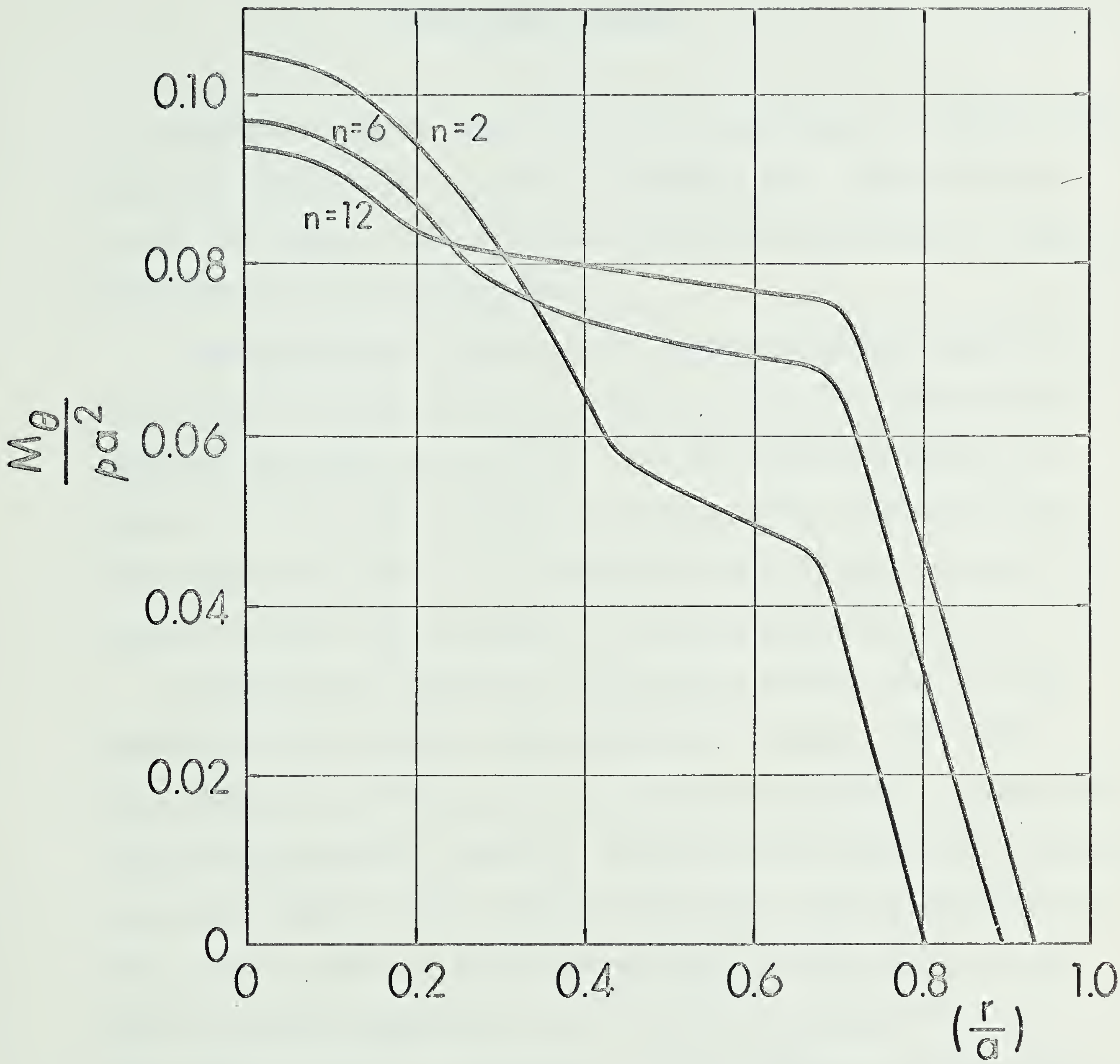

FIG. 4.3

CREEP PROBLEM

BENDING MOMENT M_θ vs. RADIUS

CHAPTER V

CONCLUDING REMARKS

Venkatraman and Hodge [16] have presented a solution for the stationary creep problem in Chapter IV. That solution is not the same as the solution found in this thesis. This discrepancy is now discussed.

Venkatraman and Hodge solved both the simply supported and built-in circular plate problems using the constitutive equation derivable from (4.2). In the former problem the stress solution was assumed to lie in stress regimes OB and BOC (referred to Fig. 3.2) whereas in the latter problem the stress solution was assumed to lie in regimes OB and OD.

Consider the region of the built-in plate whose outer boundary is the circle along which M_r is zero. The free body diagram for this portion of the plate which is supported entirely by vertical shear is identical with that for a simply supported plate. This requires that the stress solution for this inner region of the clamped plate be the same as that for the simply supported plate and the velocity solutions must differ only by a constant. This necessary requirement cannot be met using the stress profiles assumed by Venkatraman and Hodge. Since a stress profile consisting of regions OB and BOC was assumed for the simply supported plate, the stress profile for the clamped plate must also include these two stress regions.

If Venkatraman and Hodge were able to find a solution for the clamped plate using a stress profile consisting of stress regions OB and OD then the question arises can a solution be found for the simply supported plate using a stress profile consisting of stress region OB only? The answer is no and furthermore the solution found by Venkatraman and Hodge for the built-in plate can be shown to be incorrect.

If an attempt is made to solve either the visco-plastic or creep problem of a simply supported circular plate using a stress profile consisting of stress region OB only, expressions can in fact be found for w, M_r, and M_θ. Further investigation however indicates that the flow rule is violated since it can be shown that the parameter f in Tables II and V does not lie in the closed interval $0 \leqslant f \leqslant 1$ but in fact approaches infinity as r approaches a. Similarly for the clampled plate, if a stress profile consisting of stress regions OB and OD is assumed, with $r = \alpha$ being the boundary between the two regions of the plate, it can be shown that f approaches infinity as r approaches α. These solutions are thus not admissible.

Venkatraman and Hodge presented the flow rule for stress region OB as

$$M_r = M_\theta, \ \kappa_r \geqslant 0, \ \kappa_\theta \geqslant 0 \quad \text{and} \quad \kappa_r + \kappa_\theta = \left(\frac{M_r}{\mu}\right)^n G(t)$$

This is equivalent to that shown in Table V. The solution they obtained for the clamped plate violated this flow rule

since κ_r becomes negative as r approaches α.

It follows from this discussion and the uniqueness theorem in Chapter II that there is only one complete solution to each plate problem using this piecewise linear form of the surface of constant E_c. Any other kinematically admissible velocity solutions or statically admissible stress solutions which may be found can be shown by use of the extremum principles (2.10) and (2.12) not to be the real solutions.

There are a number of areas of research which could follow the work presented in this thesis. The first of course is experimental verification of the two solutions using metals near their recrystallization temperatures so that viscous effects are present. Other problems include the use of other creep laws based on the complementary function such as $E_c = \mu\lambda e^{M/\mu}$ where μ and λ are creep constants depending on the material and temperature and M is as defined in (4.1). The problem of circular plates exhibiting viscous effects which are subjected to time varying loadings such as a blast or pulse loading is also of technical importance. Solutions for this class of problems would require consideration of inertia effects.

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